Simultaneous inversion of seabed and water column sound speed profiles in range-dependent shallow-water environments

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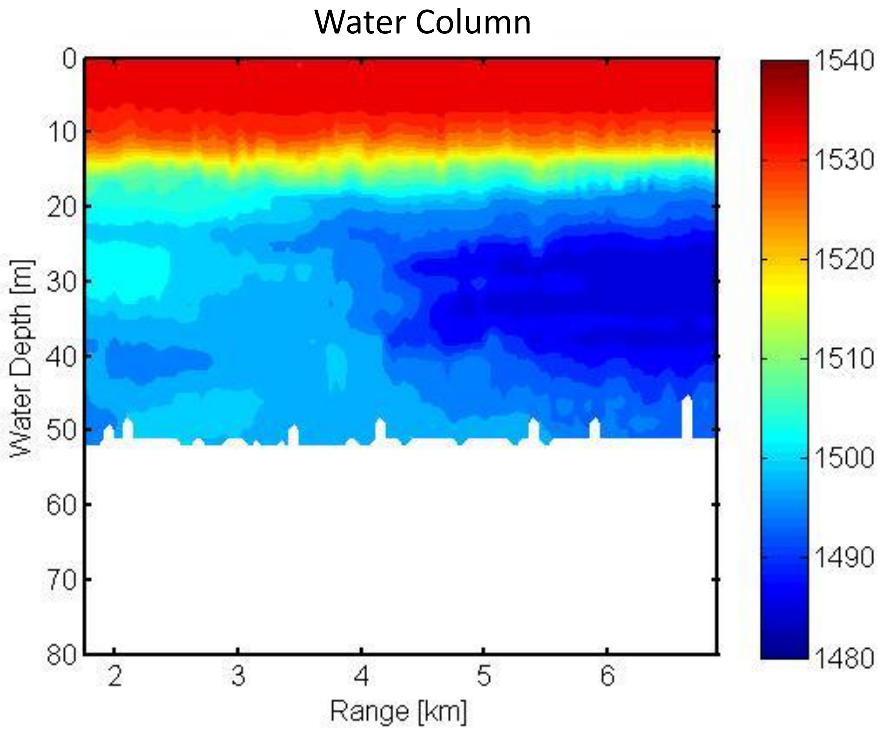
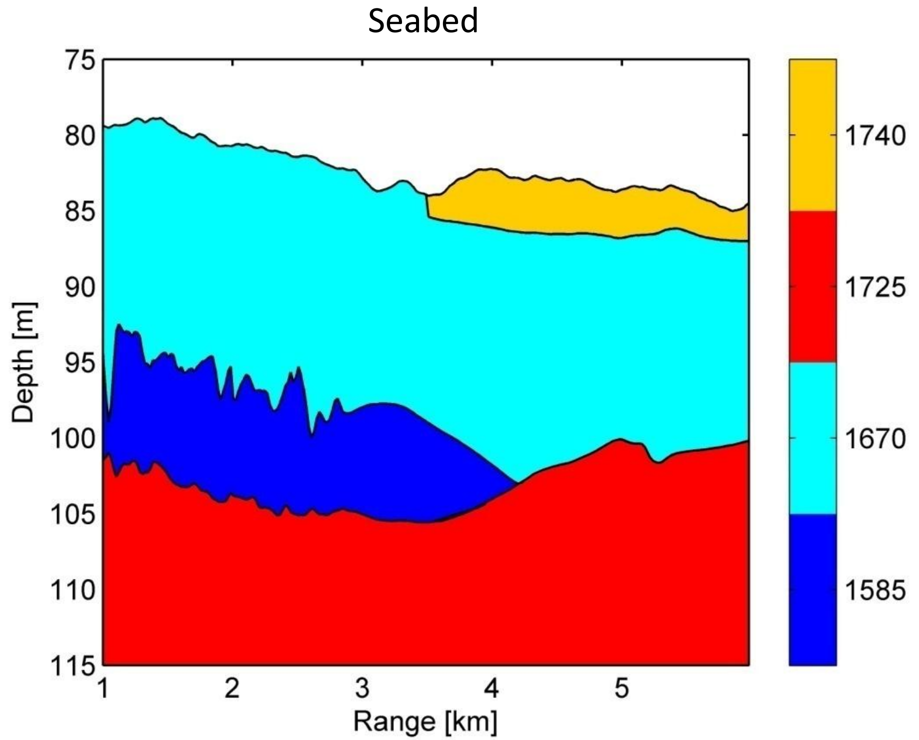
Outline

- Motivation
  - Shallow Water ’06 Experiment (SW06)
- Perturbative Inversion
  - Relates measurements of horizontal wave numbers to sounds speed
- Optimized Constraints
  - Seabed: resolve layered structure
  - Water column: inclusion of an additional constraint
  - Simultaneous inversion of seabed and water column sound speed profiles
- Evaluate the methods
  - Considering an example
  - Resolution and variance of the solution

Motivation

The approaches presented here are motivated by data from the ``Shallow Water '06'' (SW06) experiment which took place on the New Jersey shelf area of the North Atlantic in the summer of 2006. This environment is characterized by high spatial variability of both the water column and seabed.

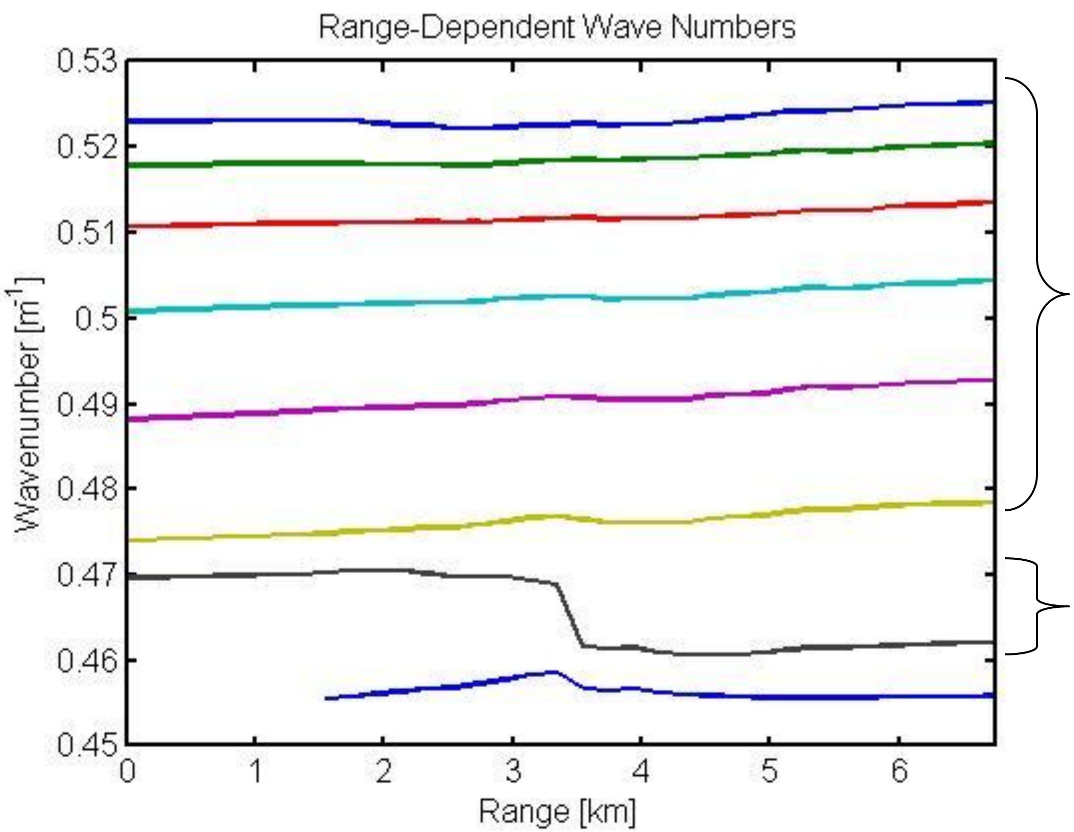




Wave Number Estimation

The Hankel Transform using the far field approximation

$$g(k_r; z, z_0) = \frac{e^{i\pi/4}}{\sqrt{2\pi k_r}} \int_{-\infty}^{\infty} p(r; z, z_0)\sqrt{r}e^{-ik_r r}dr$$



Modes one through six have increasing values which is primary caused by a decrease in the water column sound speed profile

Mode seven is a resonant mode sensitive to the low speed layer in the seabed

Perturbative Inversion

A relation between a perturbation to sound speed and a perturbation to horizontal wave numbers is formulated from the depth separated normal mode equation:

$$\Delta k_n = \frac{1}{k_n}\int_0^\infty \rho^{-1}(z) Z_n^2(z) k^2(z) \frac{\Delta c(z)}{c_0(z)}\, dz$$

This equation can be written in the form of a Fredholm integral of the first kind:

$$d_i = \int_0^\infty m(z) G_i(z)\, dz \qquad i = 1, \ldots, N$$

Which can be written in matrix form as:

$$\mathbf{d} = \mathbf{G}\mathbf{m}$$

$\mathbf{d}$ is a vector representing the data

$\mathbf{G}$ is a matrix representing the forward model (kernel)

$\mathbf{m}$ is a vector representing the model parameter

Qualitative Regularization

Inversion for the Seabed Sound Speed Profile

- In previous work, concerns about stability and uniqueness of the inverse problem had been dealt with by adding a constraint to the solution such that the smoothest sound speed profile is chosen.
- However, owing to geological processes, sediments are often better described by layers having distinct properties and are not well represented by a smooth profile.
- A new way to constrain the solution which emphasizes the layered structure of the sediment is accomplished using **qualitative regularization**.

Originally presented by the author at 154th Meeting of the ASA in New Orleans on November 27.

Approximate Equality Constraints

Inversion for the Water Column Sound Speed Profile

- To address the need for detailed information about the range-dependence of the water column, perturbative inversion is applied to estimate water column sound speed profiles.
- However, the wave number data are insufficient to determine water column properties in some portions of the waveguide.
- This issue is addressed by application of **approximate equality constraints** which force the solution to be close to likely values at prescribed locations.

Originally presented by the author at 157th Meeting of the ASA in Portland on May 20.

Joint WC and Seabed Inversion

Simultaneous Inversion for both Water Column and Seabed Sound Speed Profiles

- This technique uses a synthesis of **approximate equality constraints** and **qualitative regularization.**
  - Qualitative regularization is applied to resolve the discontinuity in the sound speed profile at the seafloor as well as additional discontinuities within the seabed.
  - Approximate equality constraints are applied to portions of the water column for which the data is inadequate to determine the solution.
- The principle benefit of inverting for water column and sediment sound speed profiles at the same time is that errors in the background environment are not aliased into the solution.

Qualitative Regularization

used to constrain the seabed inverse problem

Find a solution that satisfies the data: $\mathbf{Gm} = \mathbf{d}$

and the constraint: $\mathbf{L}_q \mathbf{m} = \mathbf{0}$

The user defined operator $\mathbf{L}_q$ is given by: $\mathbf{L}_q = \mathbf{L}(\mathbf{I} - \sum_{i=1}^{r} \mathbf{q}_i \mathbf{q}_i^T)$

where $\mathbf{L}$ is a discrete version of the differential operator $\dfrac{d^n}{d\mathbf{x}^n}$

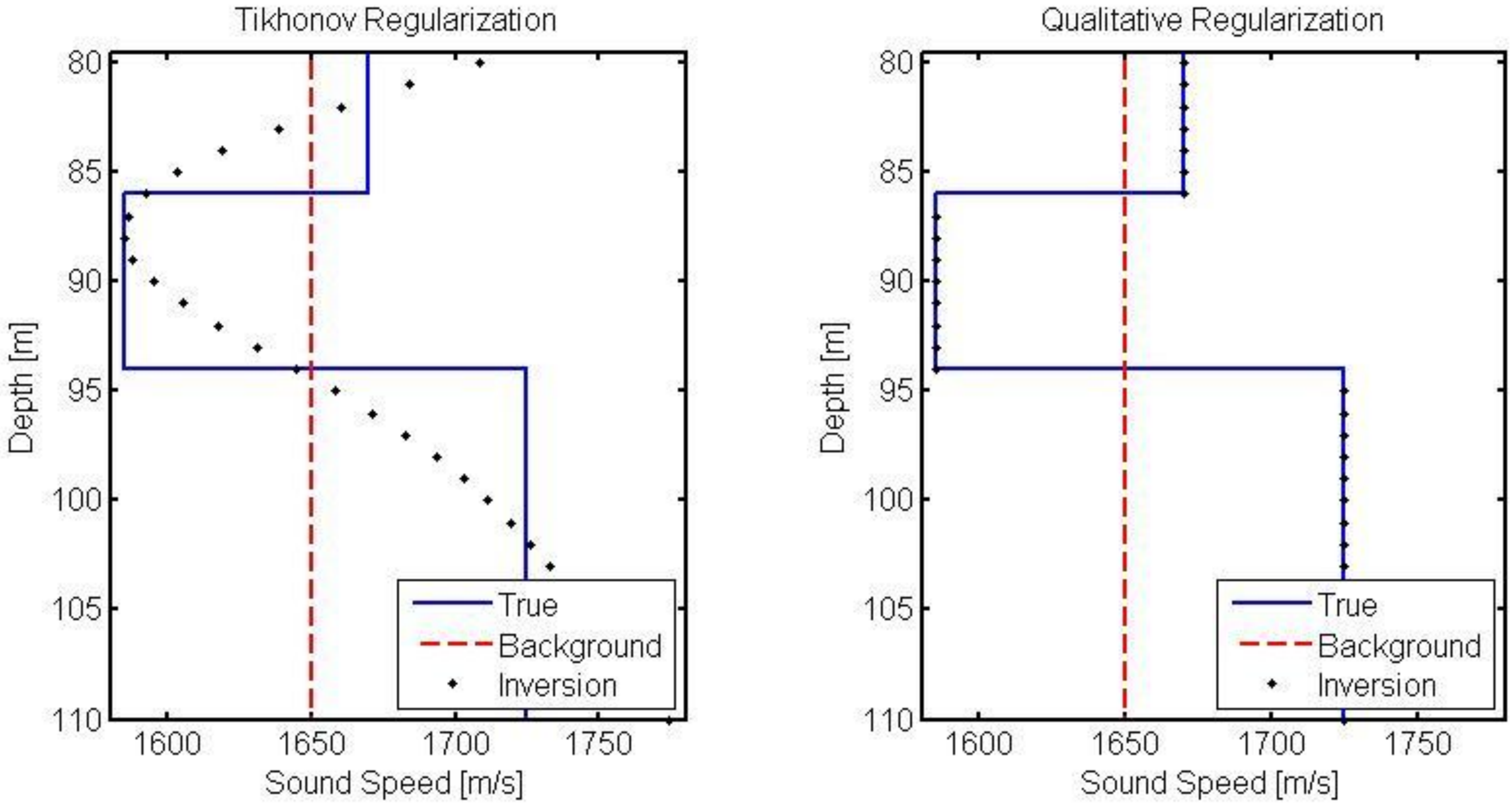
and the set $\{\mathbf{q}_i\}_{i=1}^{r}$ is on orthogonal basis for $\mathbf{Q}$.

Assigning a weight to the constraint solving simultaneously: $\begin{pmatrix} \mathbf{G} \\ \mathbf{W}\,\mathbf{L} \end{pmatrix} \mathbf{m} = \begin{pmatrix} \mathbf{d} \\ \mathbf{0} \end{pmatrix}$

Assume: $\mathbf{W}^T \mathbf{W} = \lambda^2 \mathbf{I}$

The solution is given by: $\hat{\mathbf{m}} = (\mathbf{G}^T \mathbf{G} + \lambda^2 \mathbf{L}_q^T \mathbf{L}_q)^{-1} \mathbf{G}^T \mathbf{d}$

Comparison: Tikhonov and Qualitative Regularization



Approximate Equality Constraints

used to constrain the water column inverse problem

Find a solution that satisfies the data: $\mathbf{Gm} = \mathbf{d}$

Relative Equality Constraint: $\mathbf{Lm} = \mathbf{0}$

Absolute Equality Constraint: $\mathbf{Am} = \boldsymbol{\alpha}$

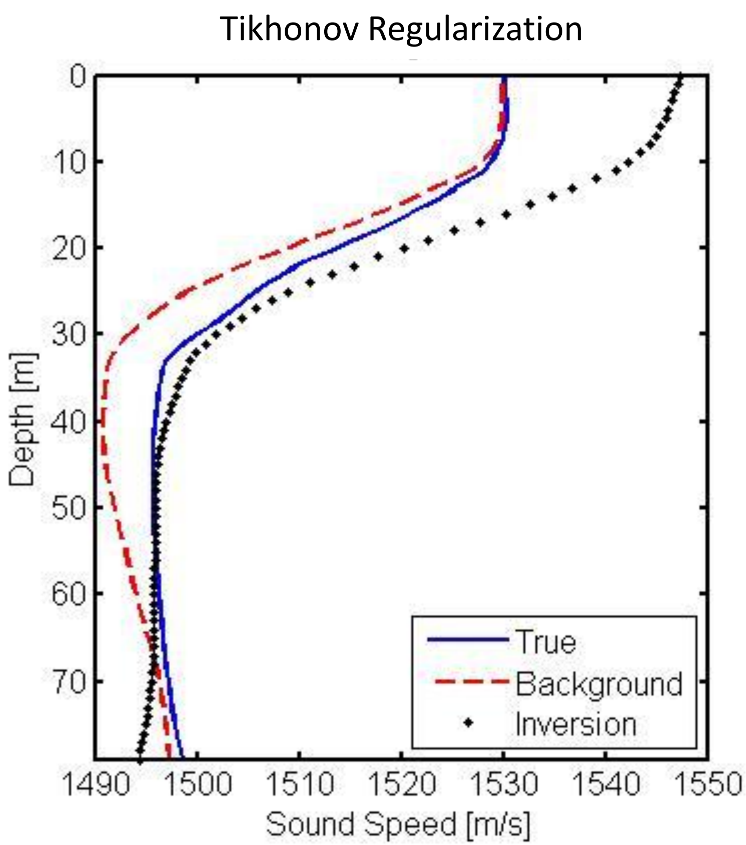
The *relative equality constraint* is the well known smoothness constraint of Tikhonov regularization.

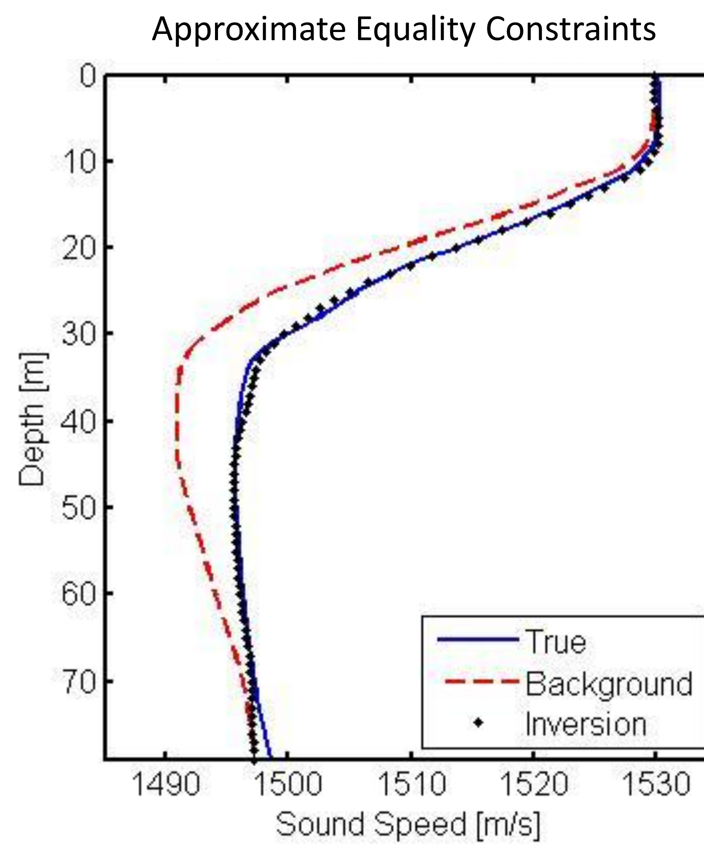
The *absolute equality constraint* is chosen to restrict perturbations from the background profile near the sea surface and seafloor.

The matrix $\mathbf{A}$ specifies where to apply the absolute equality constraint.
The vector $\boldsymbol{\alpha}$ specifies the value the solution should take at these points.

The solution is given by: $\hat{\mathbf{m}} = (\mathbf{G}^T\mathbf{G} + \lambda_1^2\mathbf{L}^T\mathbf{L} + \lambda_2^2\mathbf{A}^T\mathbf{A})^{-1}\mathbf{G}^T\mathbf{d}$

Comparison: Tikhonov Regularization and Approximate Equality Constraints





Joint Inversion Scheme

Synthesis of qualitative regularization and approximate equality constraints to estimate water column and seabed sound speed profiles simultaneously.

Qualitative Regularization

$$\hat{\mathbf{m}} = (\mathbf{G}^T\mathbf{G} + \boxed{\lambda^2 \mathbf{L}_q^T \mathbf{L}_q})^{-1}\mathbf{G}^T\mathbf{d}$$

Used to resolve the discontinuity in the sound speed profile at the seafloor as well as discontinuities in the seabed

Approximate Equality Constraints

$$\hat{\mathbf{m}} = (\mathbf{G}^T\mathbf{G} + \lambda_1^2 \mathbf{L}^T\mathbf{L} + \boxed{\lambda_2^2 \mathbf{A}^T\mathbf{A}})^{-1}(\mathbf{G}^T\mathbf{d})$$

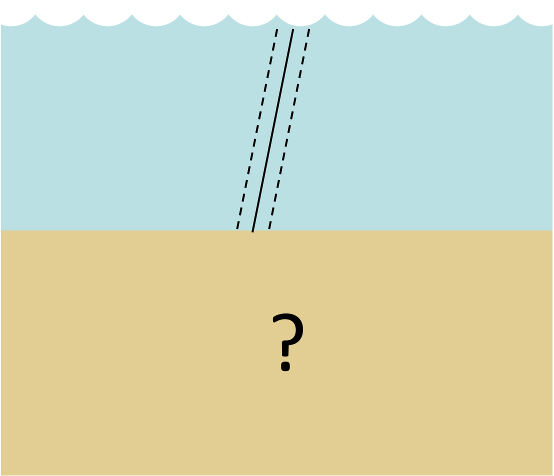
Used constrain the solution for the water column near the sea surface and seafloor where the data alone is insufficient to determine the solution

Synthesis of Qualitative Regularization and Approximate Equality Constraints

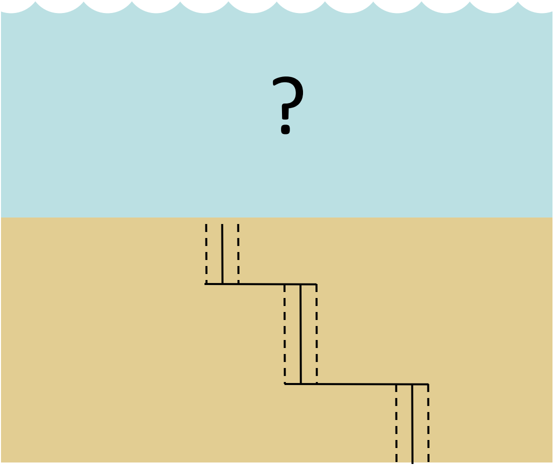
$$\hat{\mathbf{m}} = (\mathbf{G}^T\mathbf{G} + \boxed{\lambda_1^2 \mathbf{L}_q^T \mathbf{L}_q} + \boxed{\lambda_2^2 \mathbf{A}^T\mathbf{A}})^{-1}(\mathbf{G}^T\mathbf{d})$$

Robustness of the Algorithms

- Effect of incorrect assumptions about the water column when inverting for the sediment sound speed profile

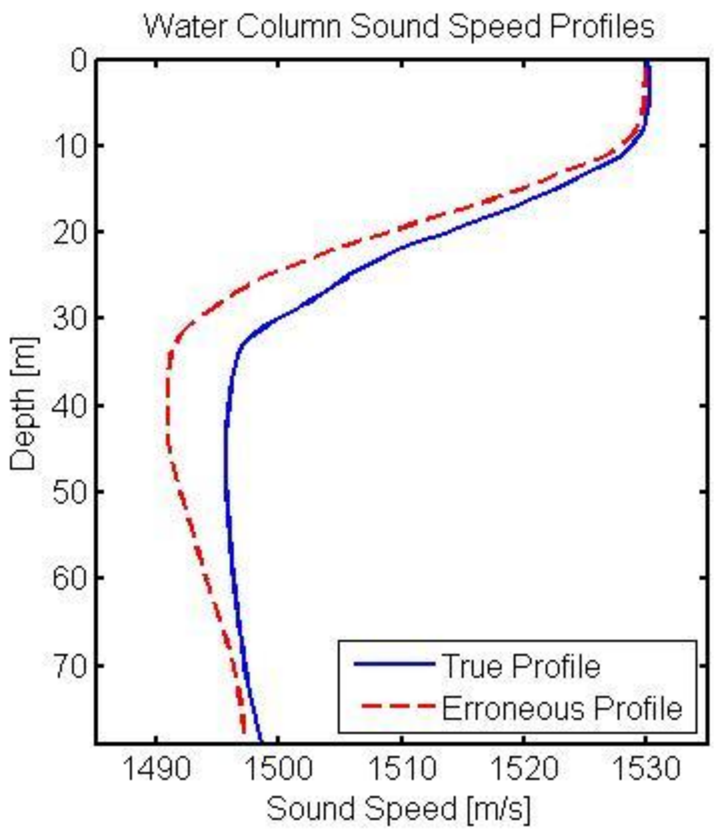


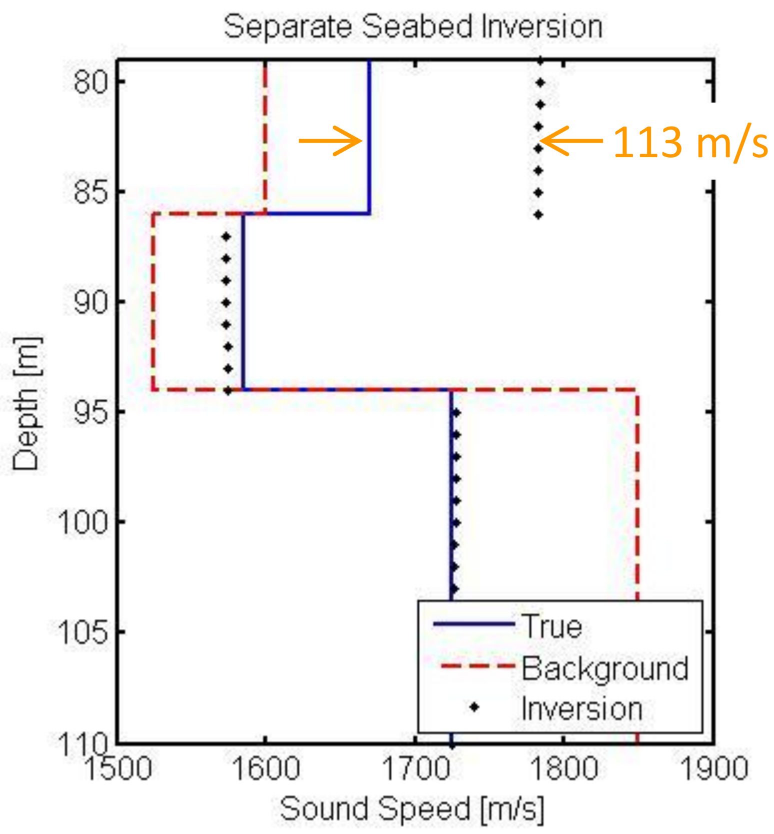
- Effect of incorrect assumptions about the seabed when inverting for the water column sound speed profile



Separate Seabed Inversion

Poor knowledge of the water column sound speed profile causes errors in the solution for the seabed sound speed profile.





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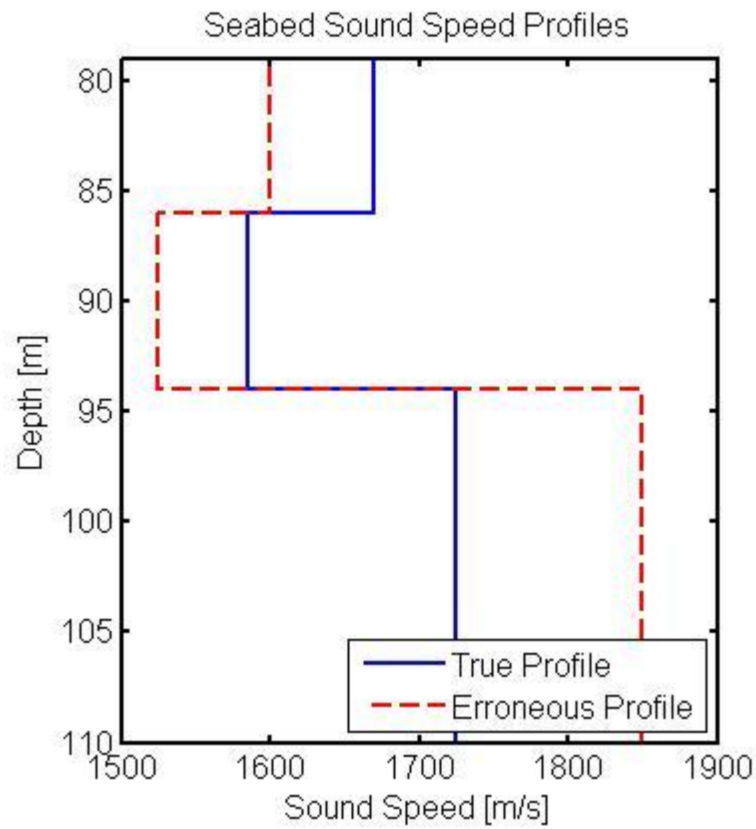
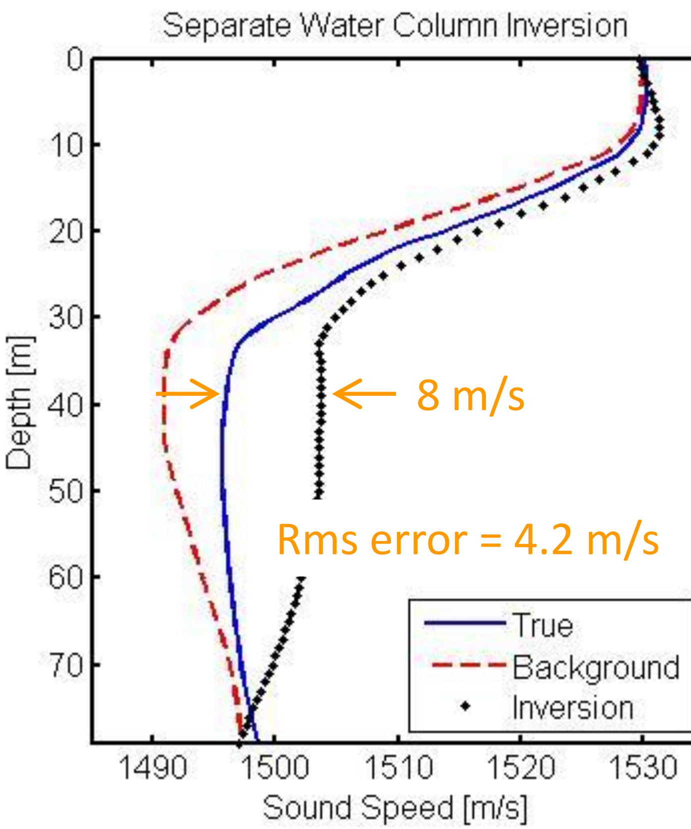
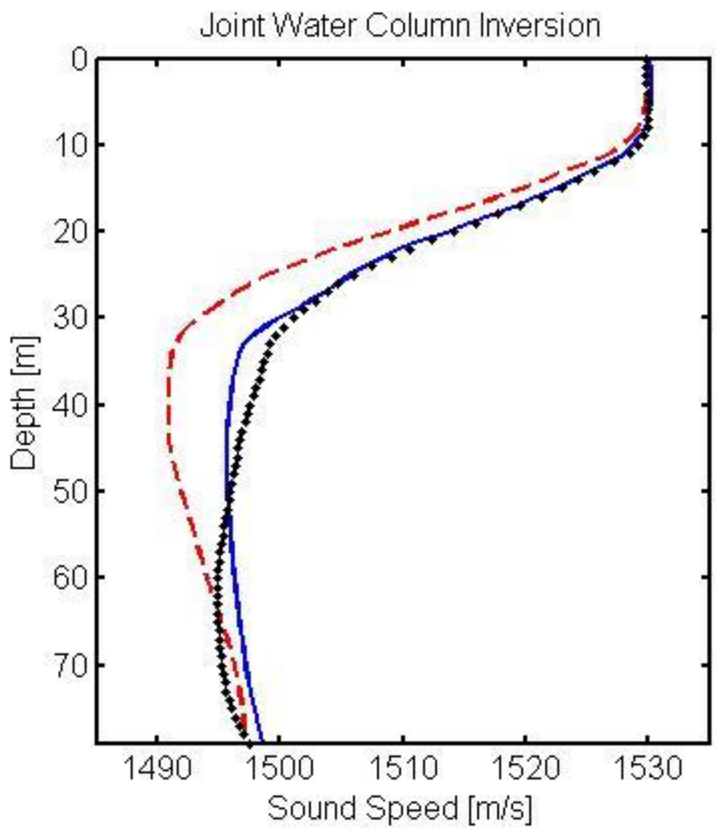
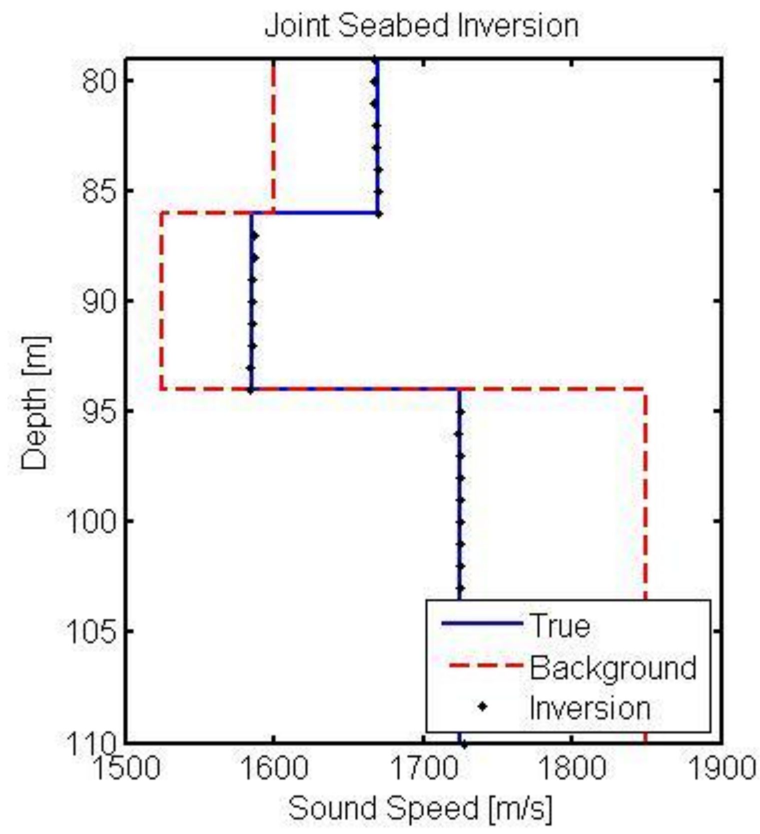




Joint WC and Seabed Inversion

Primary benefit of the joint inversion scheme is that the solution does not suffer from erroneous assumptions about the other profile.





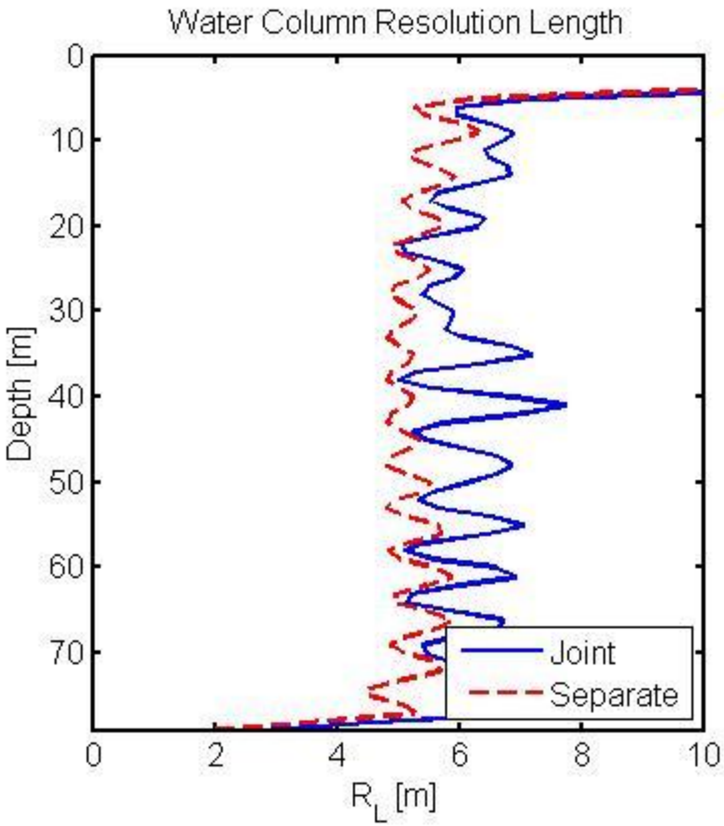
Model Resolution

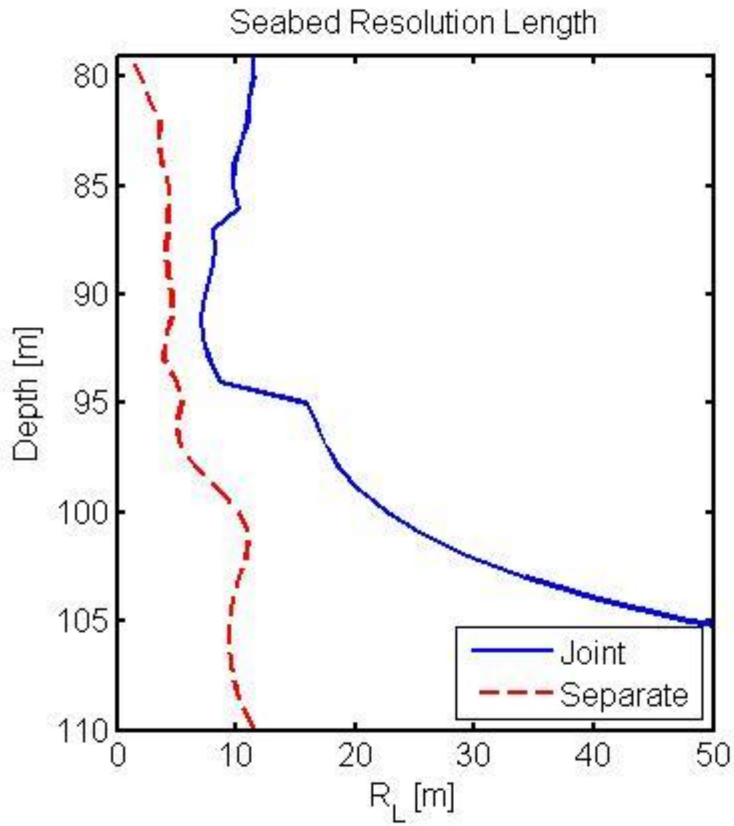
Model resolution matrix

$$\mathbf{R}_m = \mathbf{G}^T(\mathbf{G}\mathbf{G}^T)^{-1}\mathbf{G}$$

Resolution length

$$\mathbf{R}_{\mathbf{L},i} = \frac{\sum_{j=1}^{N}(\mathbf{R}_m)_{ij}^2}{(\mathbf{R}_m)_{ii}^2}$$

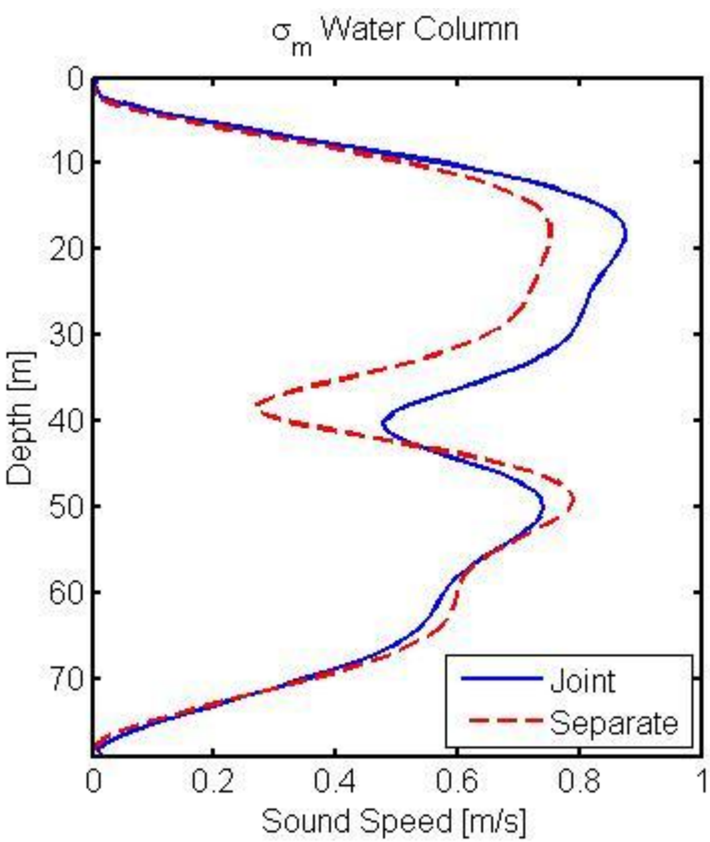


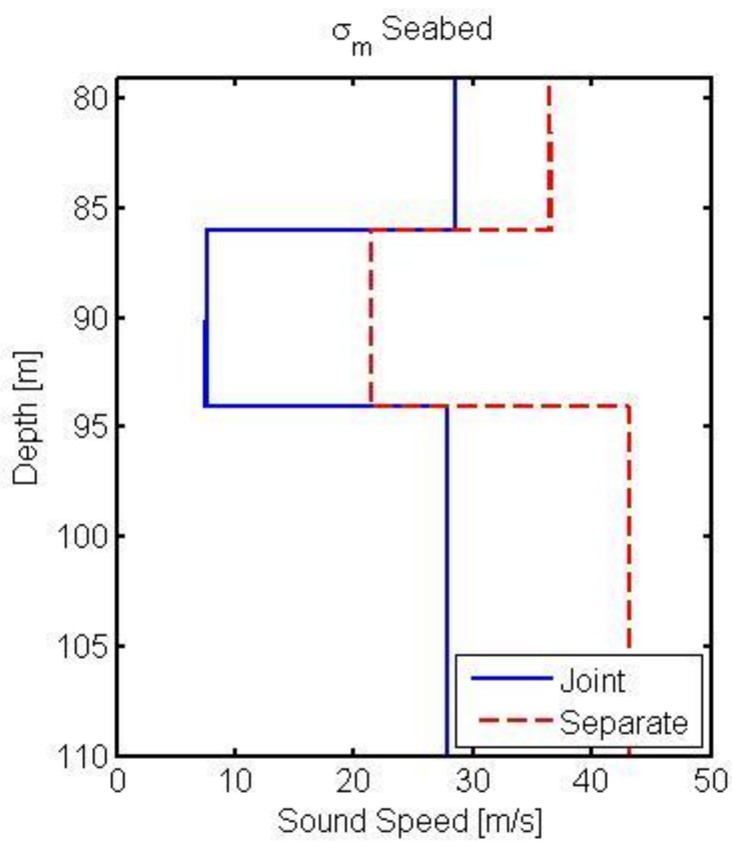


Model Variance

Monte Carlo Error Propagation

$$\mathbf{d} + \mathbf{n}_i = \mathbf{G}\mathbf{m}_i \quad i = 1, 2, \ldots, N \quad \Rightarrow \quad \sigma_m(z) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[m_i(z) - \bar{m}(z)\right]^2}$$





Summary

- Simultaneous inversion of seabed and water column sound speed profiles
  - Use of **qualitative regularization** to resolve the discontinuity in the sound speed profile at the seafloor as well as additional discontinuities within the seabed
  - Use of **approximate equality constraints** to constrain the solution where the data was inadequate to determine the solution
  - Obtain accurate solutions for both seabed and water column sound speed profiles when neither profile is known
- Resolution and Variance
  - Separate inversion schemes provide a solution with higher resolution
  - Joint inversion scheme provides a solution with lower variance that is more stable

Thank You

Questions?

Ill-Posed Problem

This is the problem we wish to solve:

$$\mathbf{d} = \mathbf{G}\mathbf{m}$$

Data [Nx1]

Kernel [NxM]

Parameters [Mx1]

Solution to the Unconstrained Inverse Problem:

$$\hat{\mathbf{m}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{d}$$

The unconstrained least squares solution is an *ill-posed problem*.

Ill-posed inverse problems are those that when there are small errors on the data can create large deviations in the solution. There may be infinitely many least squares solutions to this problem, so the solution is found by choosing one that has some characteristic of the expected solution.

Wave Number Estimation

The Hankel Transform Pair using the far field approximation

$$p(r;z,z_0)=\frac{e^{-i\pi/4}}{\sqrt{2\pi r}}\int_{-\infty}^{\infty} g(k_r;z,z_0)\sqrt{k_r}e^{ik_r r}dk_r$$

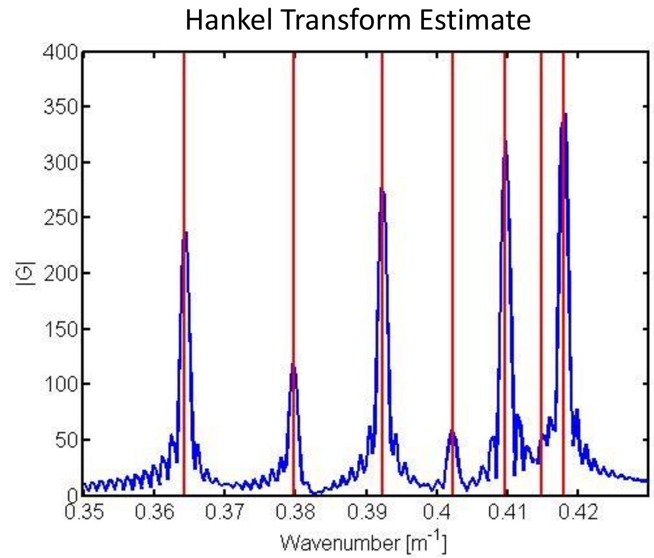
$$g(k_r;z,z_0)=\frac{e^{i\pi/4}}{\sqrt{2\pi k_r}}\int_{-\infty}^{\infty} p(r;z,z_0)\sqrt{r}e^{-ik_r r}dr$$

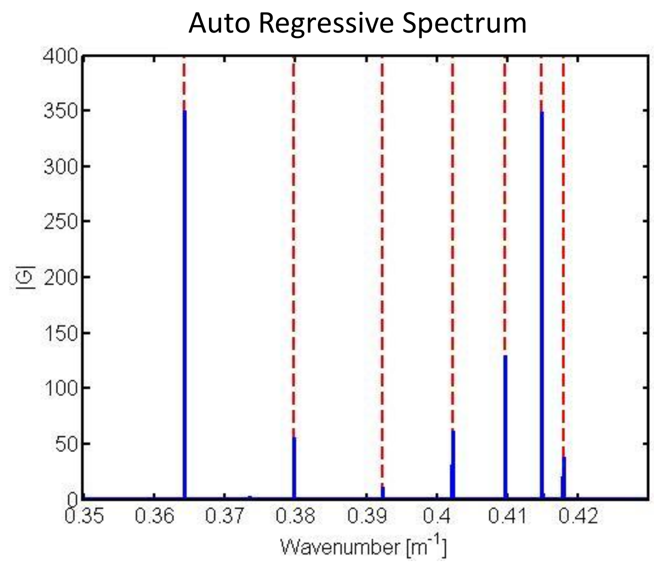
The Short-Time Fourier Transform (STFT)

$$g(k_r;\hat{r},z,z_0)=\frac{e^{i\pi/4}}{\sqrt{2\pi k_r}}\int_{-\infty}^{\infty} w_L(r;\hat{r})p(r;z,z_0)\sqrt{r}e^{-ik_r r}dr$$

Auto Regression (AR)

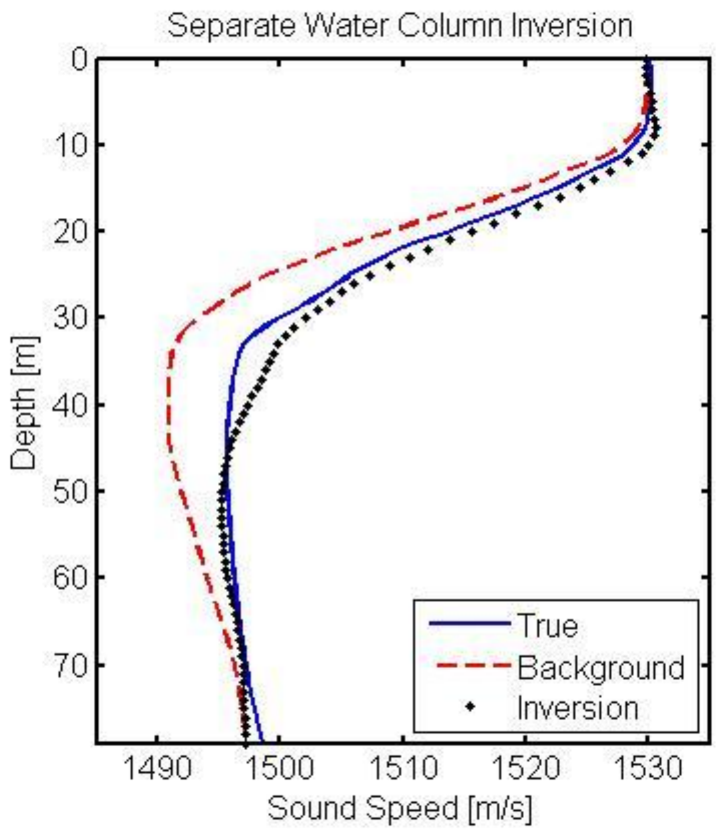
$$x_n=\sum_{k=1}^{p}a_k x_{n-k} \longrightarrow P_{AR}=\frac{\sigma^2 T}{\left|1+\sum_{k=1}^{p}a_k e^{-i2\pi fkT}\right|^2}$$

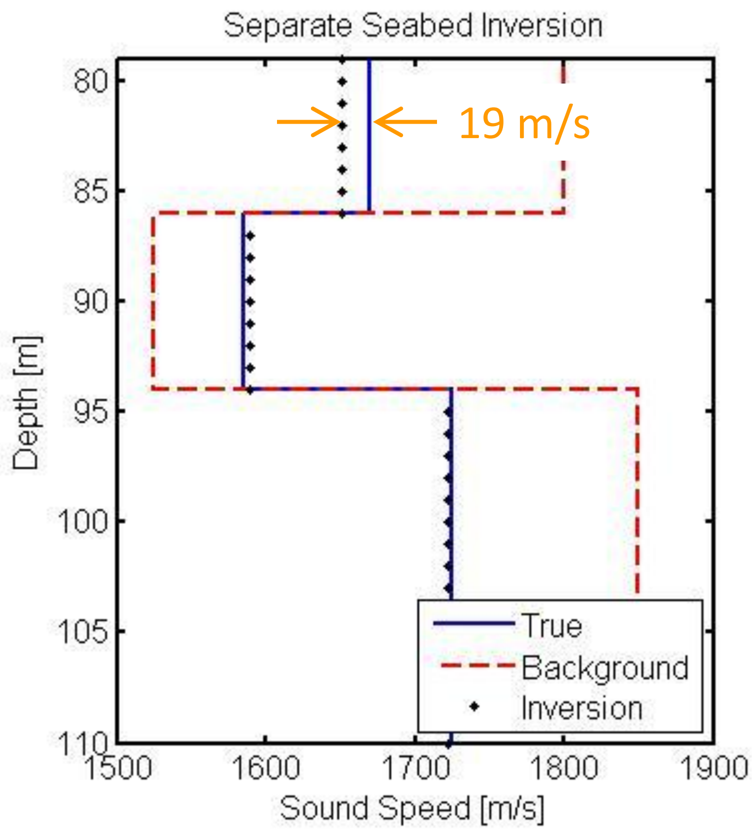




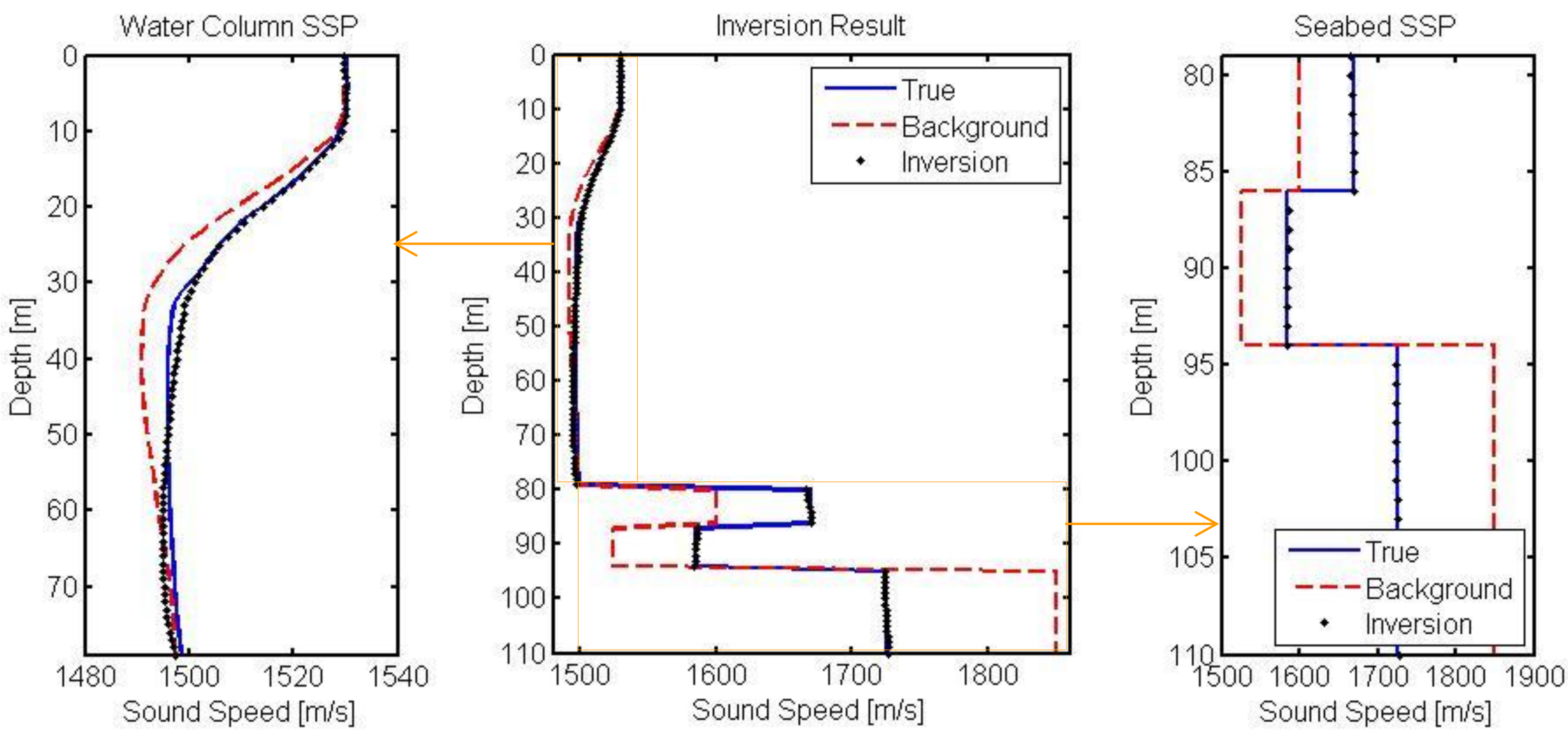
Separate Seabed Inversion

An improved solution for the seabed sound speed profile can be obtained using the inverted water column sound speed profile.



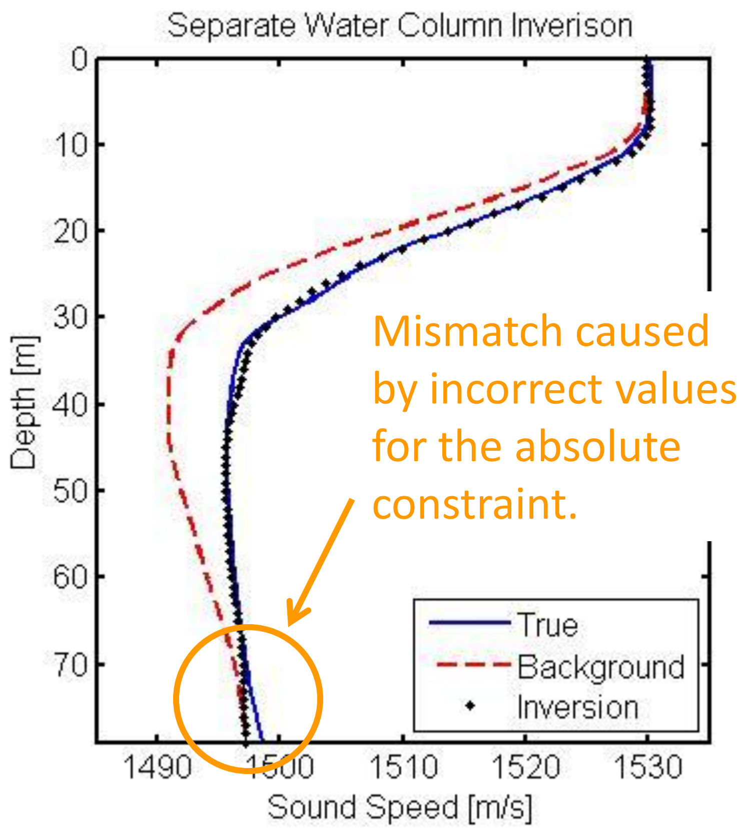


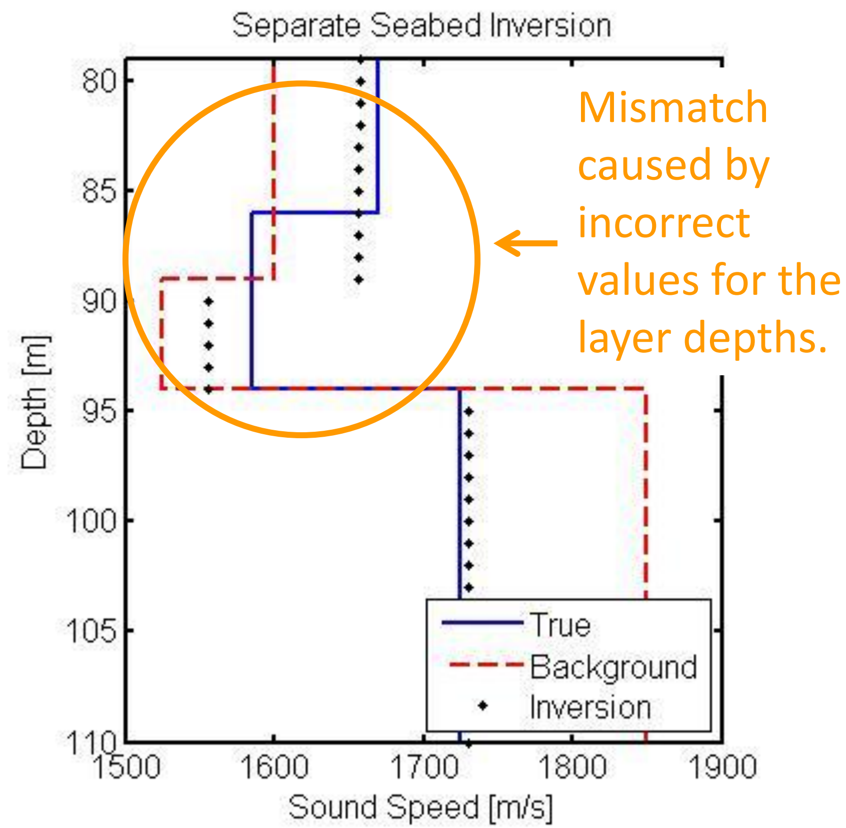
Joint Inversion Scheme



Faulty *a priori* Information

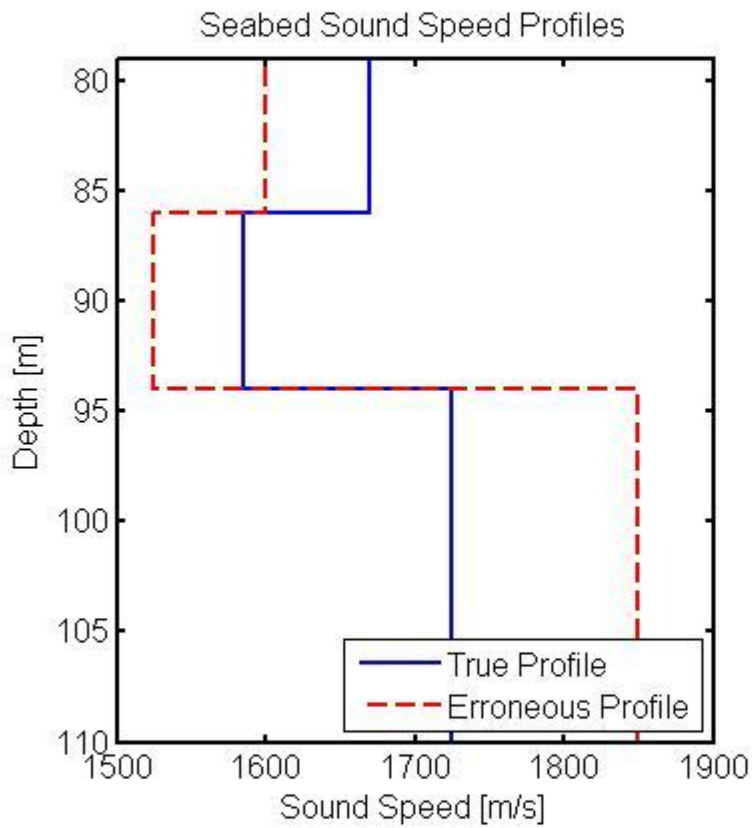
Faulty *a priori* information will cause errors in the solution: incorrect values for the absolute constraints and layer depths will be reflected in the solutions.

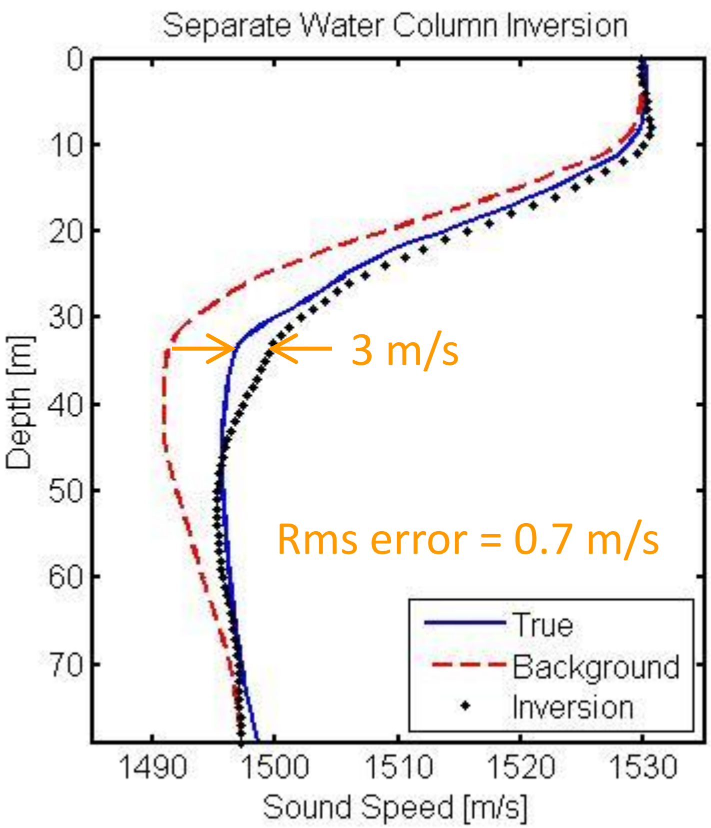




Separate Water Column Inversion

The solution for the water column sound speed profile can be improved by using a subset of low order wave number data.





Resolution and Variance

It is essential to investigate how well the solution to the inverse problem represents the true model parameter values.

- **Model resolution** is examined to characterize the bias of the generalized inverse.
  - It shows how closely the solution matches the true model given exact data and assuming the known parameters are exact.

- **Model variance** is studied to understand the effect of data errors.
  - It is a measure how inaccuracies in the data affect the solution.

Model resolution and model variance are calculated for each of the water column and seabed inverse problems using both the separate and joint techniques.

Model Resolution

The continuous inverse problem is inherently underdetermined. All estimates of the model parameters must be given in terms of local averages.

$$\hat{m}(z_0) = \int_0^D R(z, z_0) m(z) dz$$

Model resolution matrix

$$\mathbf{R}_m = \mathbf{G}^T (\mathbf{G}\mathbf{G}^T)^{-1} \mathbf{G}$$

For perfect resolution:

$$\mathbf{R}_m = \mathbf{I}$$

Else:

$\mathbf{R}_m$ is symmetric matrix that describes how the inverse smears out the true model.

Resolution length

$$\mathbf{R}_{\mathbf{L},i} = \frac{\sum_{j=1}^{N} (\mathbf{R}_m)_{ij}^2}{(\mathbf{R}_m)_{ii}^2}$$

For perfect resolution:

$$\mathbf{R}_{\mathbf{L},i} = 1 \forall i$$

Else:

$$\mathbf{R}_{\mathbf{L},i} > 1 \forall i$$

Model Variance

Model variance accounts for the effect of data errors on the solution. It is a measure how inaccuracies in the data affect the solution.

Monte Carlo Error Propagation:

$$\mathbf{d} + \mathbf{n}_i = \mathbf{G}\mathbf{m}_i \qquad i = 1, 2, ..., N$$

The noise vector was zero mean Gaussian distributed with $\sigma_d = 5\times 10^{-4}$

The empirical estimate of the covariance matrix

$$\mathbf{Cov} = \frac{\mathbf{A}^T\mathbf{A}}{N} \quad \text{where} \quad \mathbf{A} = \mathbf{m}_i^T - \bar{\mathbf{m}}_i^T$$

Only convergent solutions were considered

Error bars: standard deviation of the model estimates

$$\sigma_m(z) = \sqrt{diag(\mathbf{Cov})}$$

Solution Stability

A measure of the instability of the solution to an inverse problem is given by its condition number.

$$\mathbf{cond}(\mathbf{G}) = \frac{S_1}{S_k}$$

where S_1 and S_k are the largest and smallest singular values of G.

The condition number can be interpreted as the rate at which the solution $\mathbf{m}$ will change with respect to a change in the data $\mathbf{d}$. Thus, if the condition number is large, even a small error in $\mathbf{d}$ may cause a large error in $\mathbf{m}$.

Separate Seabed Inversion	Separate Water Column Inversion	Joint Seabed and Water Column Inversion
8.86×10^{9}	2.49×10^{2}	1.90×10^{2}