Inversion for Range Dependent Water Column Sound Speed Profiles in Shallow Water

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Overview

Motivation: Geoacoustic inversion for seabed properties can be adversely affected by poor knowledge of the water column properties.

Objective: Develop an inversion scheme to estimate range dependent water column sound speed profiles.

Method: Apply *approximate equality constraints* to constrain the solution to the perturbative inversion algorithm.

Perturbative Inversion

A relation between a perturbation to sound speed in sediment and a perturbation to horizontal wavenumbers is formulated from the depth separated normal mode equation:

$$\Delta k_n = \frac{1}{k_n}\int_0^\infty \rho^{-1}(z) Z_n^2(z) k^2(z) \frac{\Delta c(z)}{c_0(z)} dz$$

This equation can be written in the form of a Fredholm integral of the first kind:

$$d_i = \int_0^\infty m(z) G_i(z) dz \qquad i = 1, \ldots, N$$

Which can be written in matrix form as:

$$\mathbf{d} = \mathbf{Gm}$$

$\mathbf{d}$ is a vector representing the data

$\mathbf{G}$ is a matrix representing the forward model

$\mathbf{m}$ is a vector representing the model parameter

Approximate Constraints

This is the problem we wish to solve:

$$\mathbf{d} = \mathbf{G}\mathbf{m}$$

Data [Nx1] Model [NxM] Parameters [Mx1]

We want to solve for $\mathbf{m}$ under these constraints:

Relative Equality Constraint	[LxM]	$\mathbf{R}\mathbf{m} = \rho$	[Lx1]
Absolute Equality Constraint	[HxM]	$\mathbf{A}\mathbf{m} =$	[Hx1]

Assigning different weights to each constraint and combining them:

$$\begin{pmatrix} \mathbf{G} \\ \mathbf{W_R}\mathbf{R} \\ \mathbf{W_A}\mathbf{A} \end{pmatrix} \mathbf{m} = \begin{pmatrix} \mathbf{d} \\ \mathbf{W_R}\rho \\ \mathbf{W_A} \end{pmatrix}$$

Approximate Constraints

The least squares solution for $\mathbf{m}$ is given by:

$$\hat{\mathbf{m}} = (\boxed{\mathbf{G}^T\mathbf{G}} + \lambda_1 \mathbf{R}^T\mathbf{R} + \lambda_2 \mathbf{A}^T\mathbf{A})^{-1}(\boxed{\mathbf{G}^T\mathbf{d}} + \lambda_1 \mathbf{R}^T\rho + \lambda_2 \mathbf{A}^T\alpha)$$

Unconstrained Inversion Problem

If the Lagrange multipliers are equal to zero the equation reduces to the well known unconstrained least squares solution:

$$\lambda_1 = \lambda_2 = 0 \longrightarrow \hat{\mathbf{m}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{d}$$

The unconstrained least squares solution is an *ill-posed problem*.

Ill-posed inverse problems are those that when there are small errors on the data can create large deviations in the solution. There maybe infinitely many least squares solutions to this problem, so the solution is found by choosing one that has some characteristic of the expected solution.

Approximate Constraints

The least squares solution for $\mathbf{m}$ is given by:

$$\hat{\mathbf{m}} = (\mathbf{G}^T\mathbf{G} + \boxed{\lambda_1 \mathbf{R}^T\mathbf{R}} + \lambda_2 \mathbf{A}^T\mathbf{A})^{-1}(\mathbf{G}^T\mathbf{d} + \boxed{\lambda_1 \mathbf{R}^T\rho} + \lambda_2 \mathbf{A}^T\alpha)$$

Relative Equality Constraints

$\mathbf{R}$ is a discrete version second differential operator and ρ is the zero vector
These conditions specify a smooth solution.

$$\mathbf{R} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & \vdots & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\rho = [\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ]^{\mathbf{T}}$$

Approximate Constraints

The least squares solution for $\mathbf{m}$ is given by:

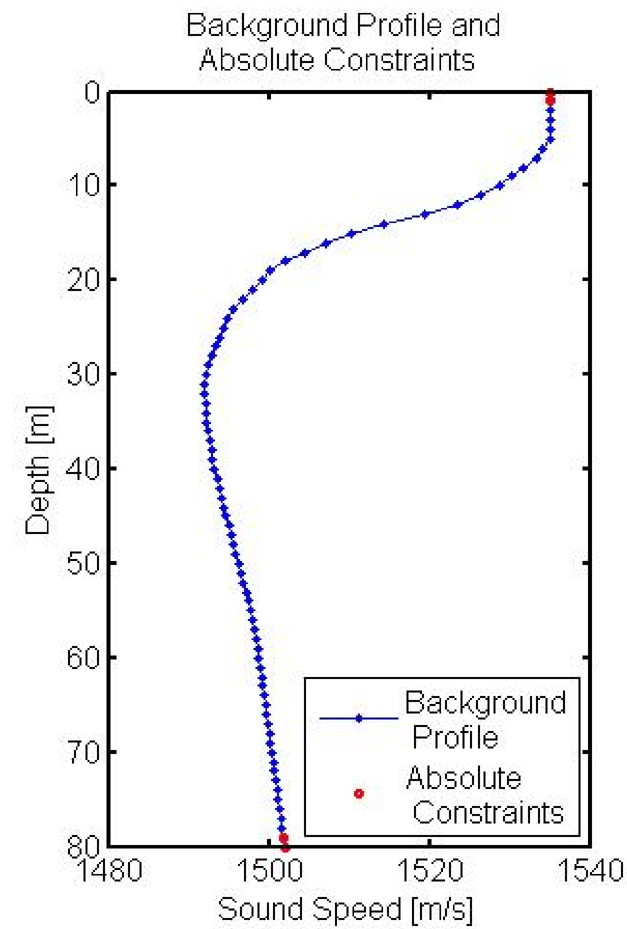
$$\hat{\mathbf{m}} = (\mathbf{G}^T\mathbf{G} + \lambda_1\mathbf{R}^T\mathbf{R} + \boxed{\lambda_2\mathbf{A}^T\mathbf{A}})^{-1}(\mathbf{G}^T\mathbf{d} + \lambda_1\mathbf{R}^T\rho + \boxed{\lambda_2\mathbf{A}^T\alpha})$$

Absolute Equality Constraints

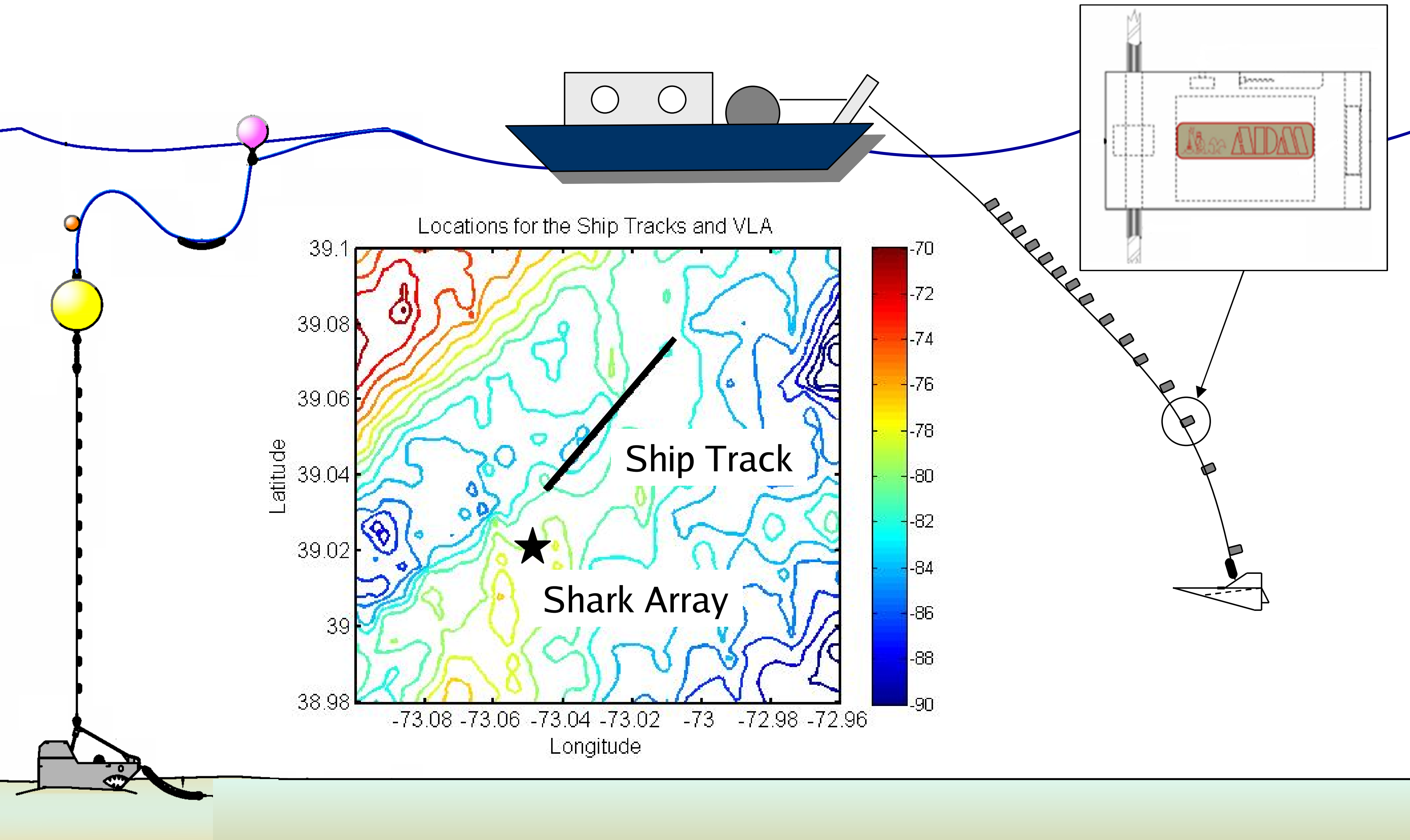
$\mathbf{A}$ and are given below. These conditions do not allow perturbations to the background at the specified locations.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [\ 0\ \ 0\ \ 0\ \ 0\ \ ]^{\mathbf{T}}$$

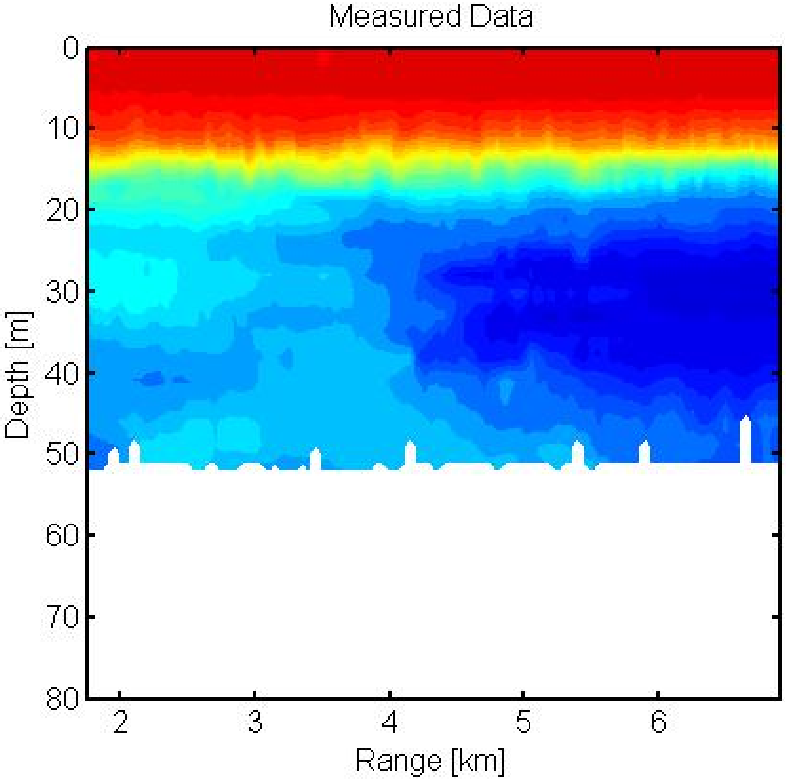


SW06 Experiment

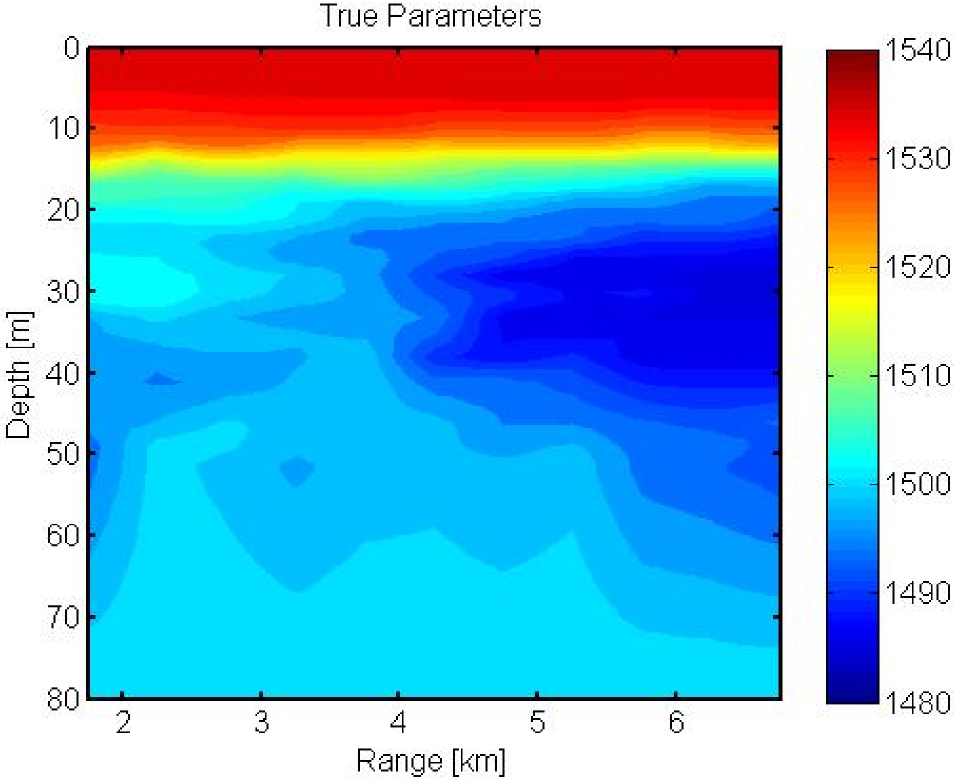


CTD Chain Measurement

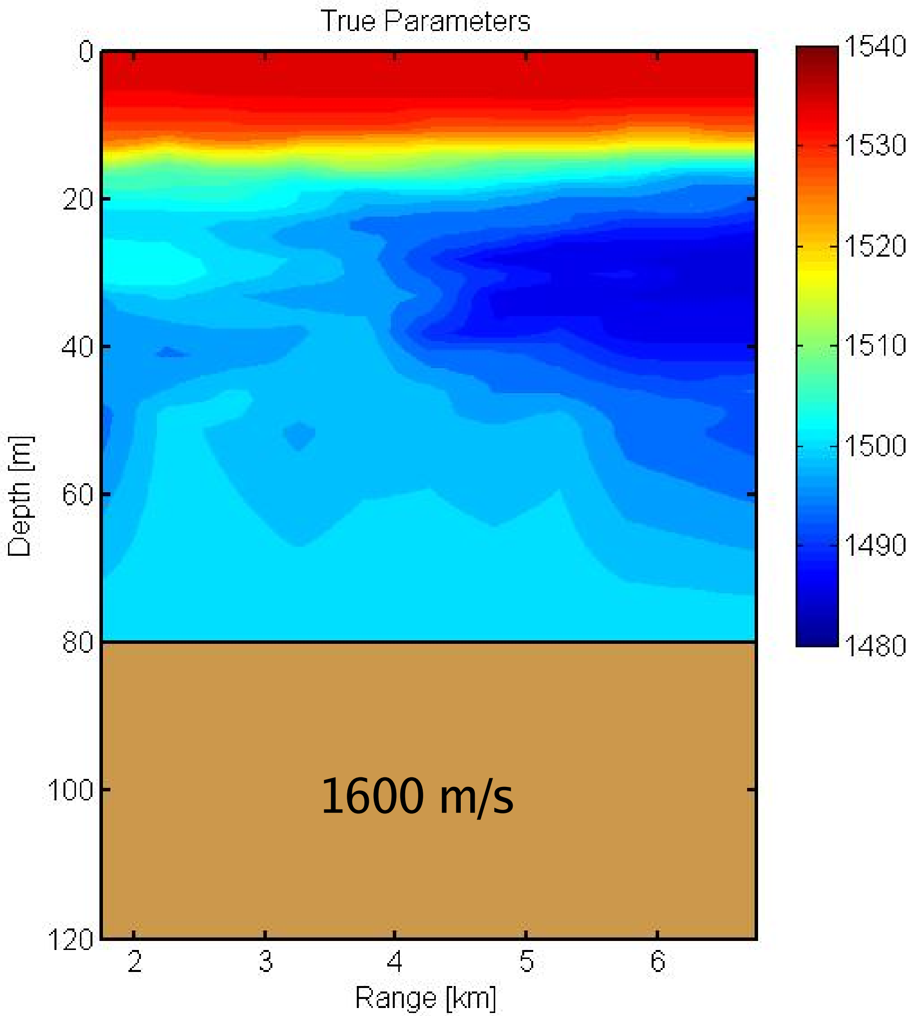
Sound speed calculated from CTD chain measurements

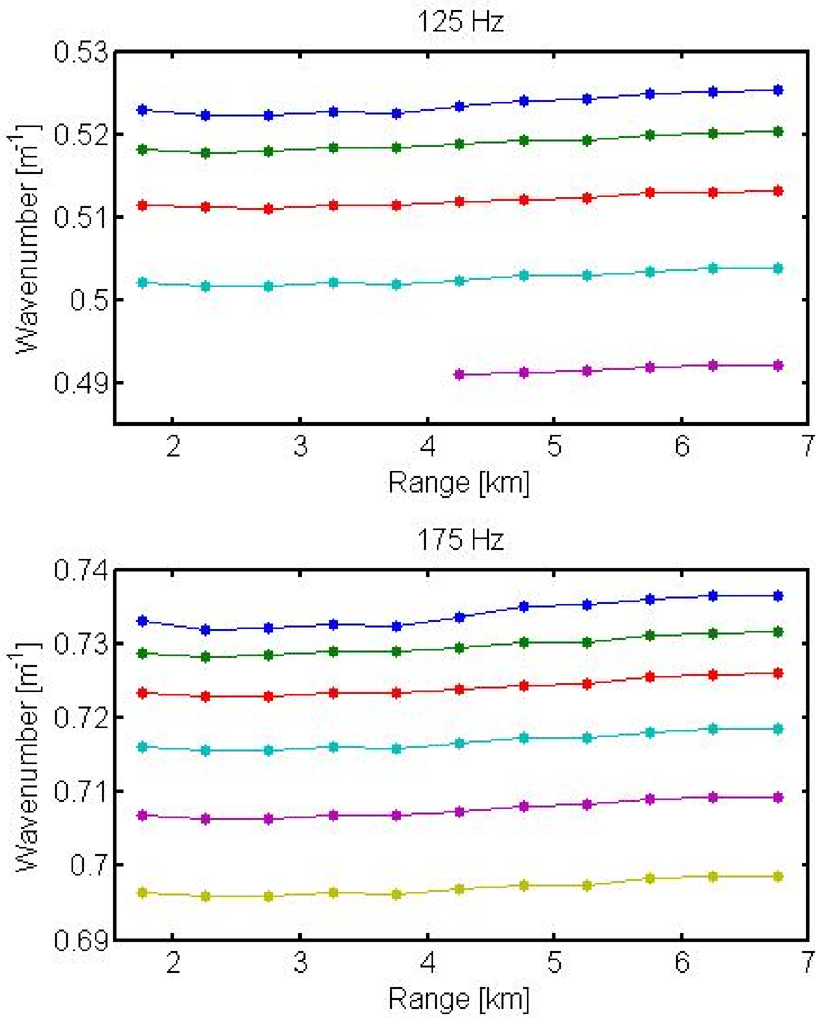


Subset of 11 measurements Interpolated to the seafloor



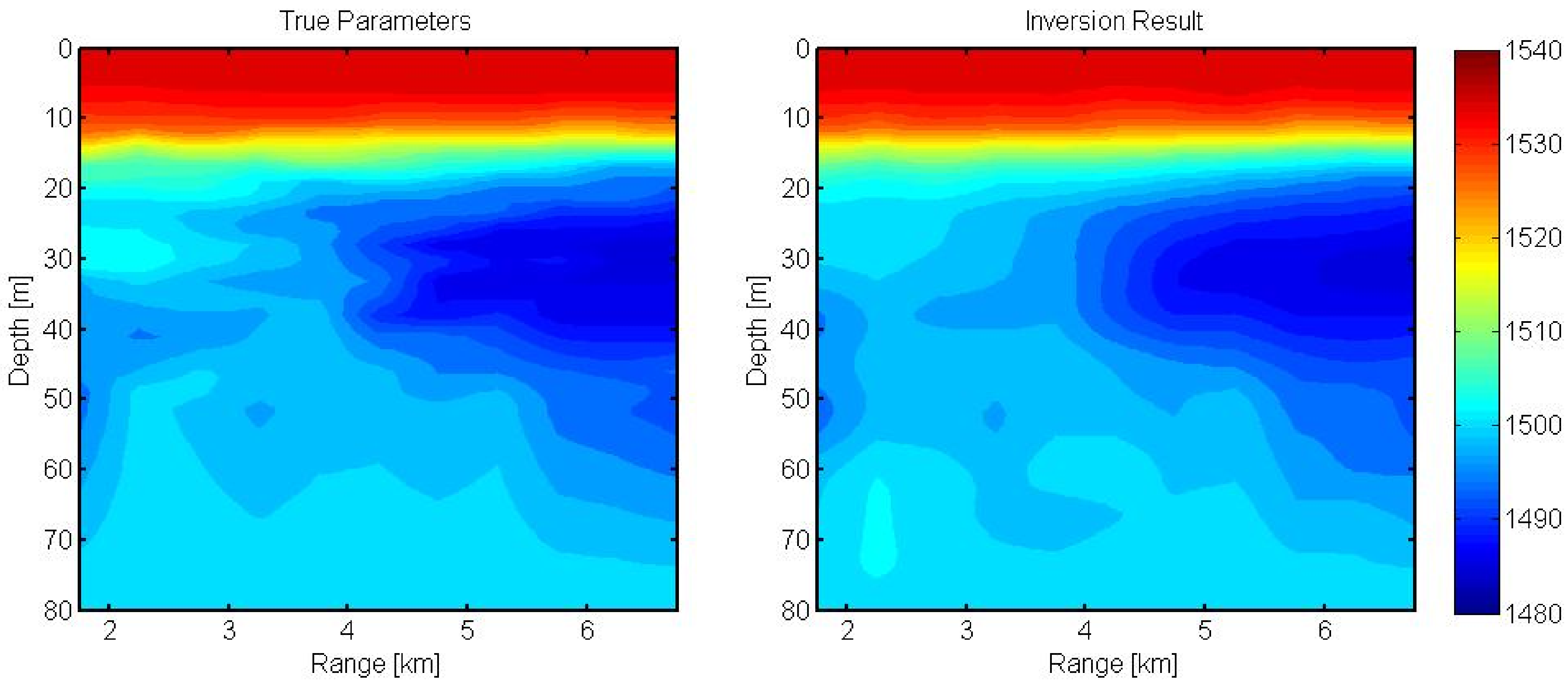
Wavenumber Data





Inversion Results

Results obtained using 125 and 175 Hz wavenumbers



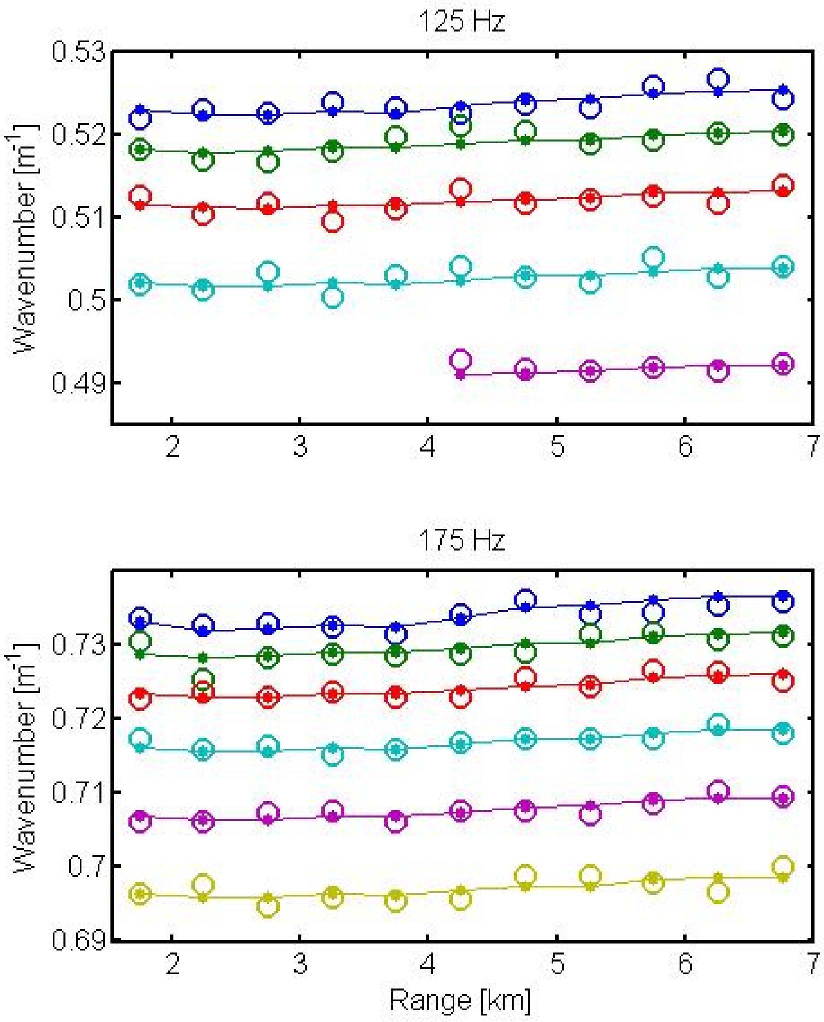
Effect of inaccurate data

In practice, the wavenumber data are not measured directly – they must be estimated from the pressure field.

Effects of the estimation process:

- Wavenumber data will be spatially averaged over some range aperture.
- Wavenumber data will not be perfectly accurate.

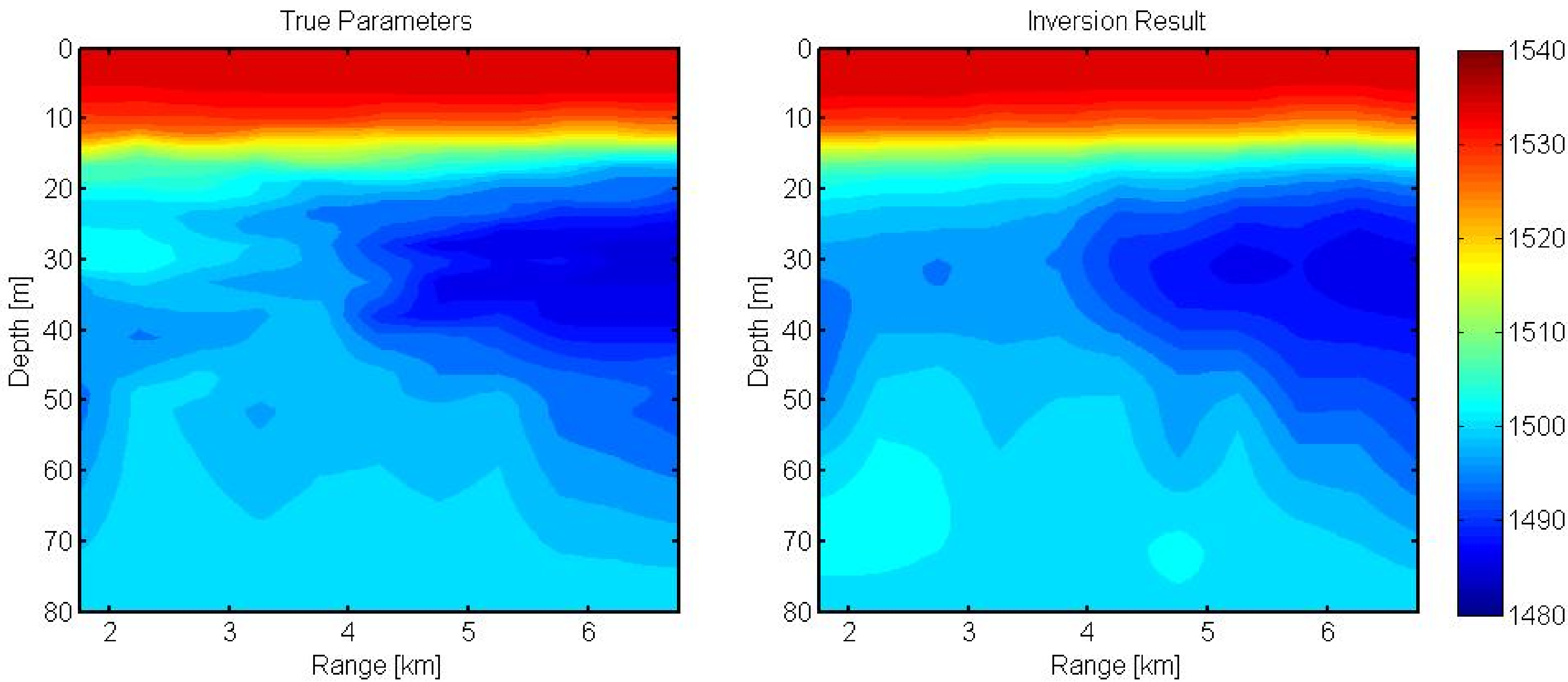
To simulate inaccurate data, zero mean Gaussian distributed noise was added to the calculated wavenumbers.



Effect of inaccurate data

Results obtained using 125 and 175 Hz wavenumbers

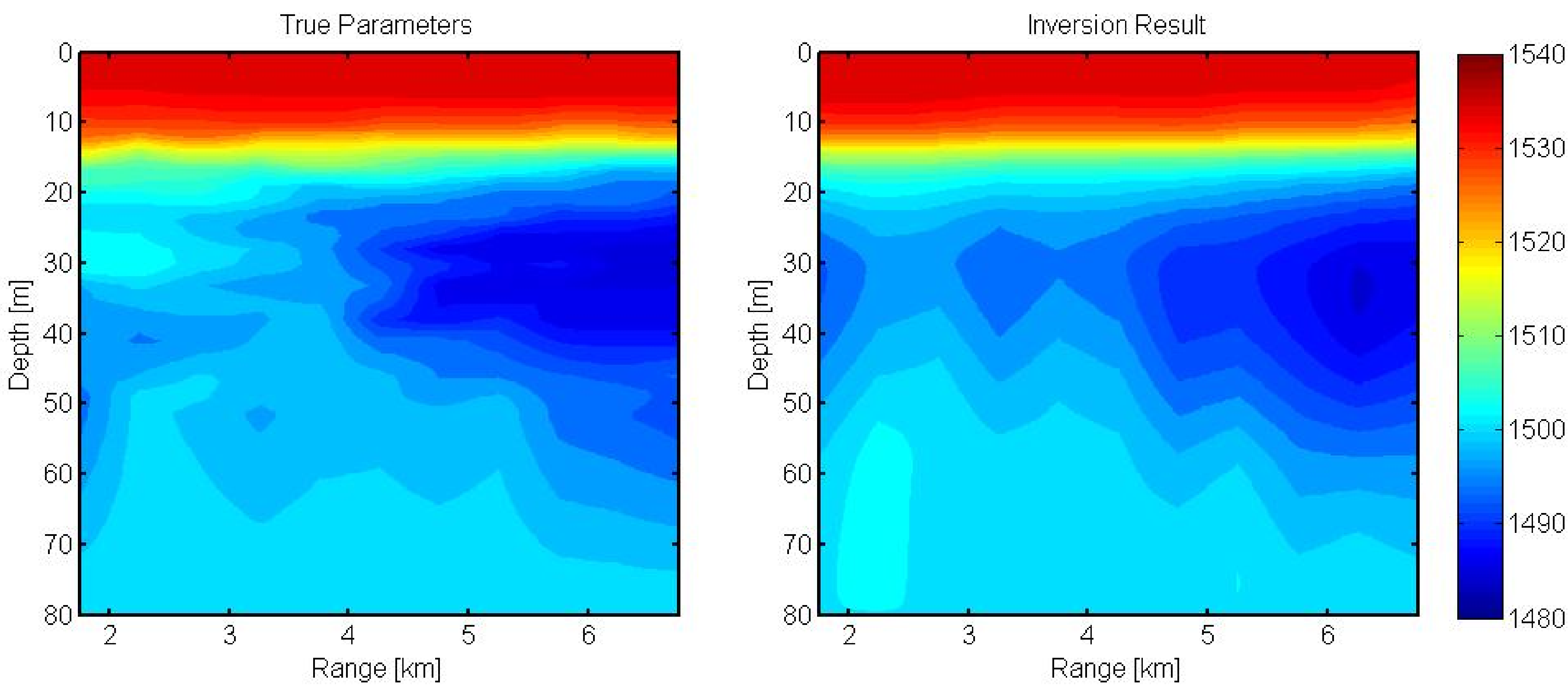
added noise: $s = 1e-4$



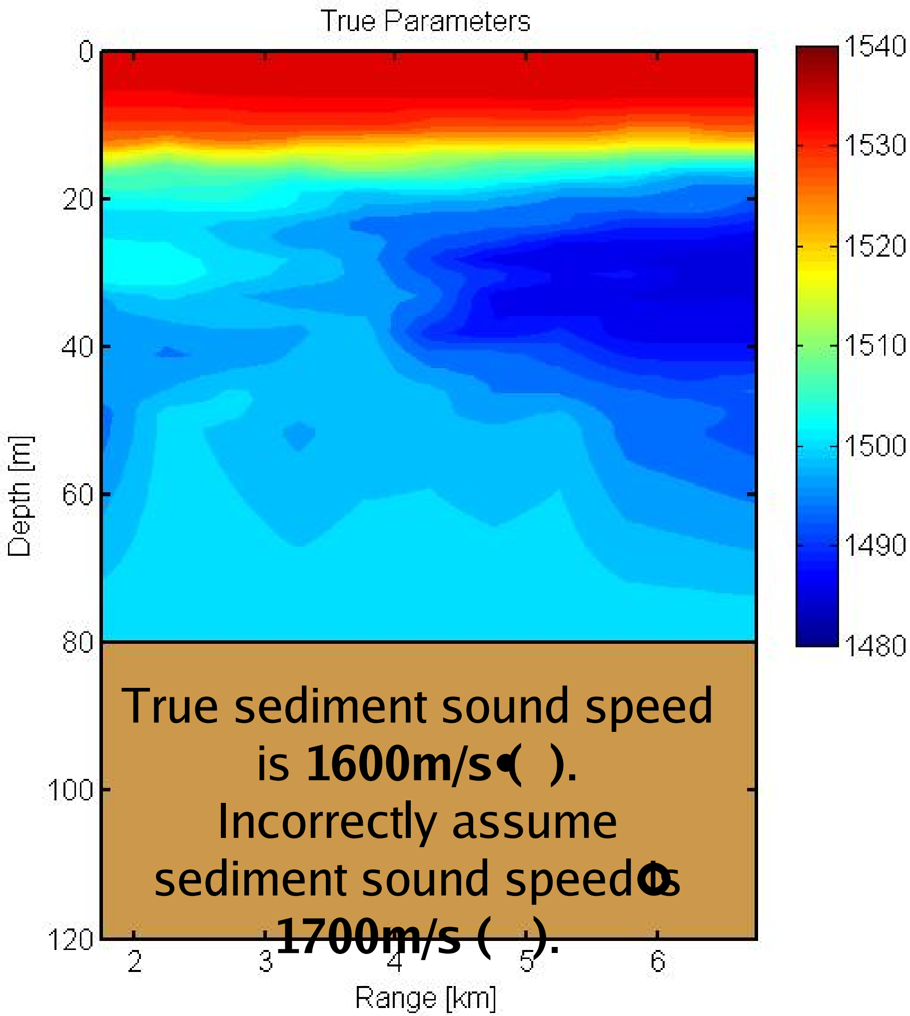
Effect of inaccurate data

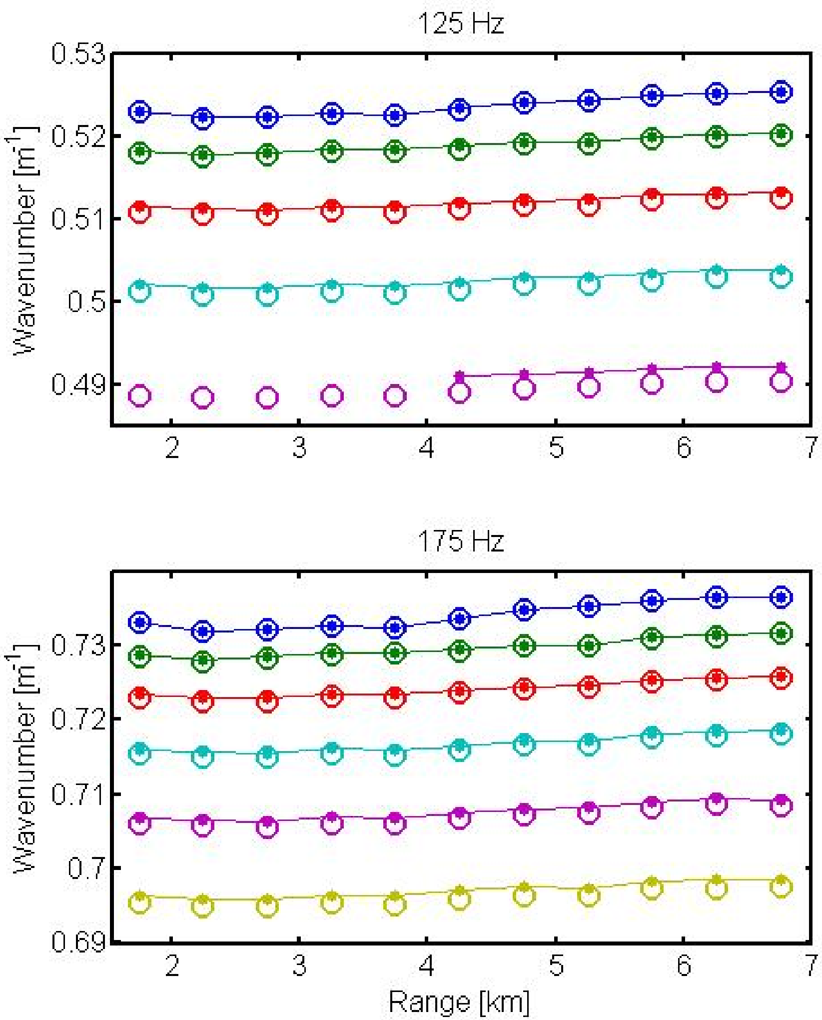
Results obtained using 125 and 175 Hz wavenumbers

added noise: $s = 1e - 3$



Effect of poor knowledge of the bottom

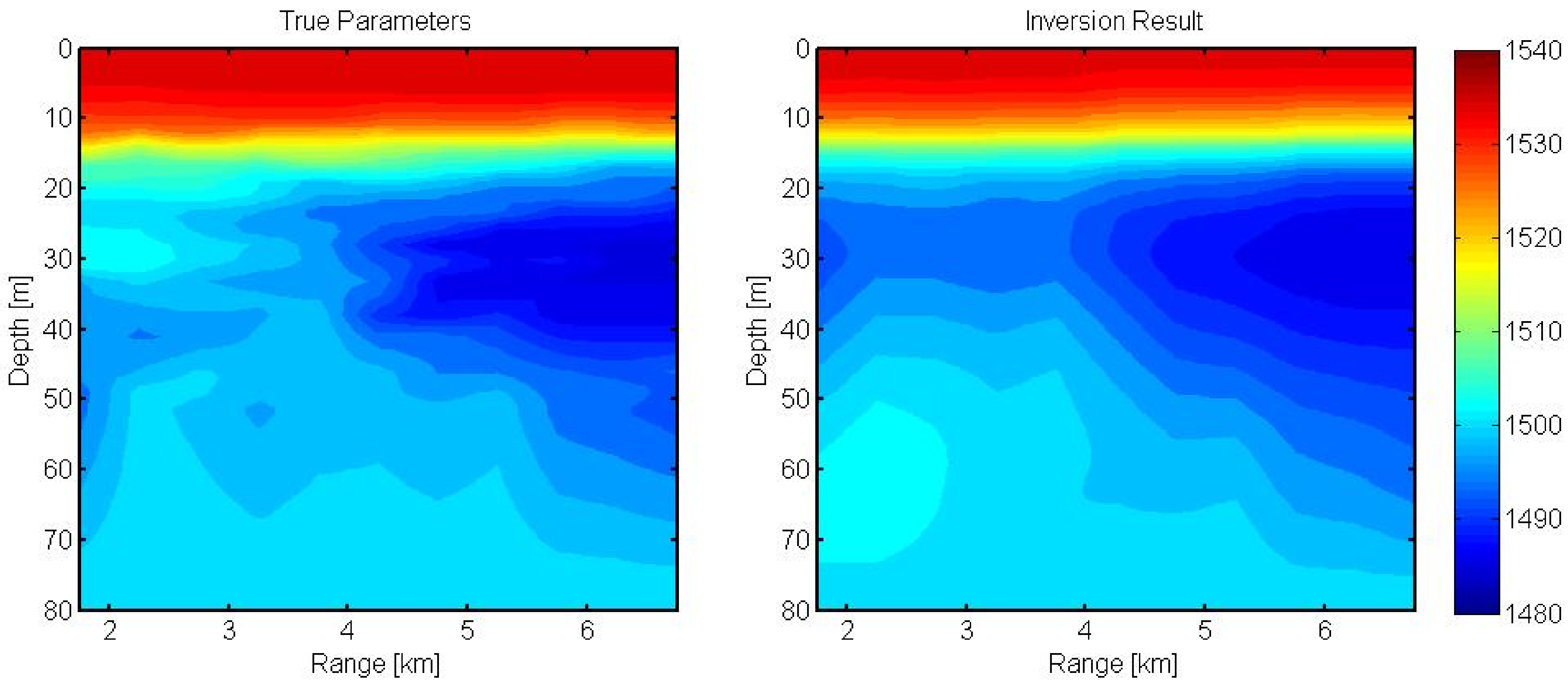




Effect of poor knowledge of the bottom

Results obtained using 125 and 175 Hz wavenumbers

Error in input sediment sound speed = 100m/s

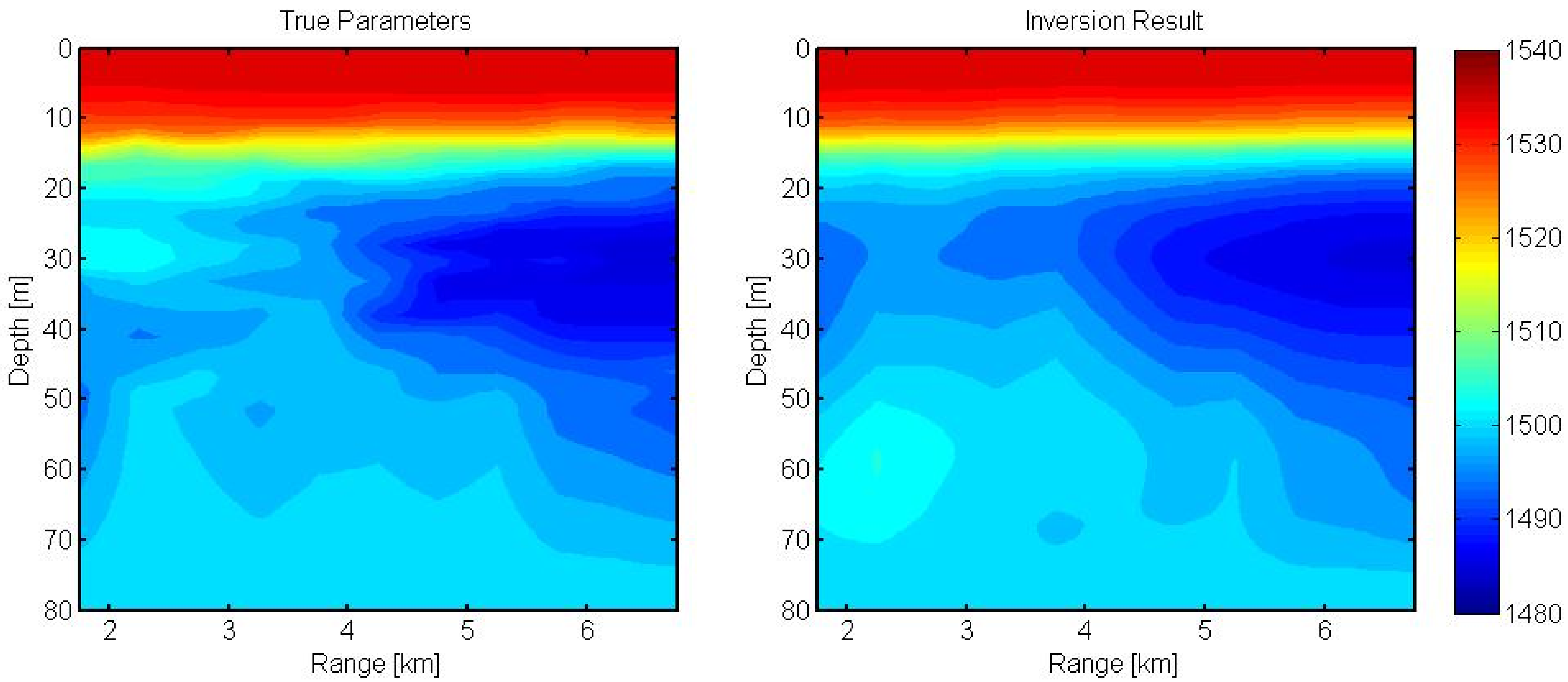


Effect of poor knowledge of the bottom

Results obtained using 125 and 175 Hz wavenumbers

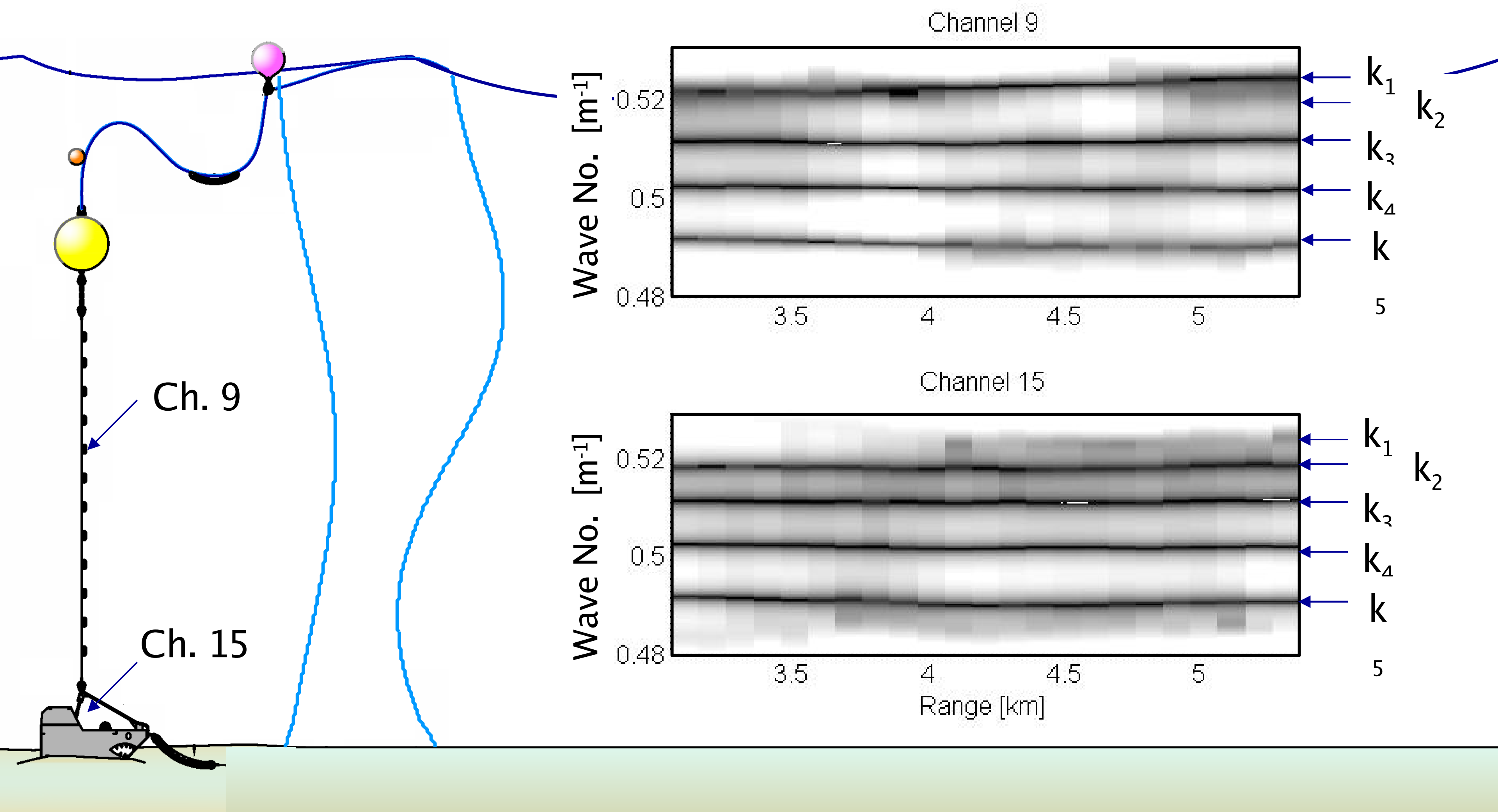
Error in input sediment sound speed = 100m/s

Using low order wavenumbers only.



Data from SW06

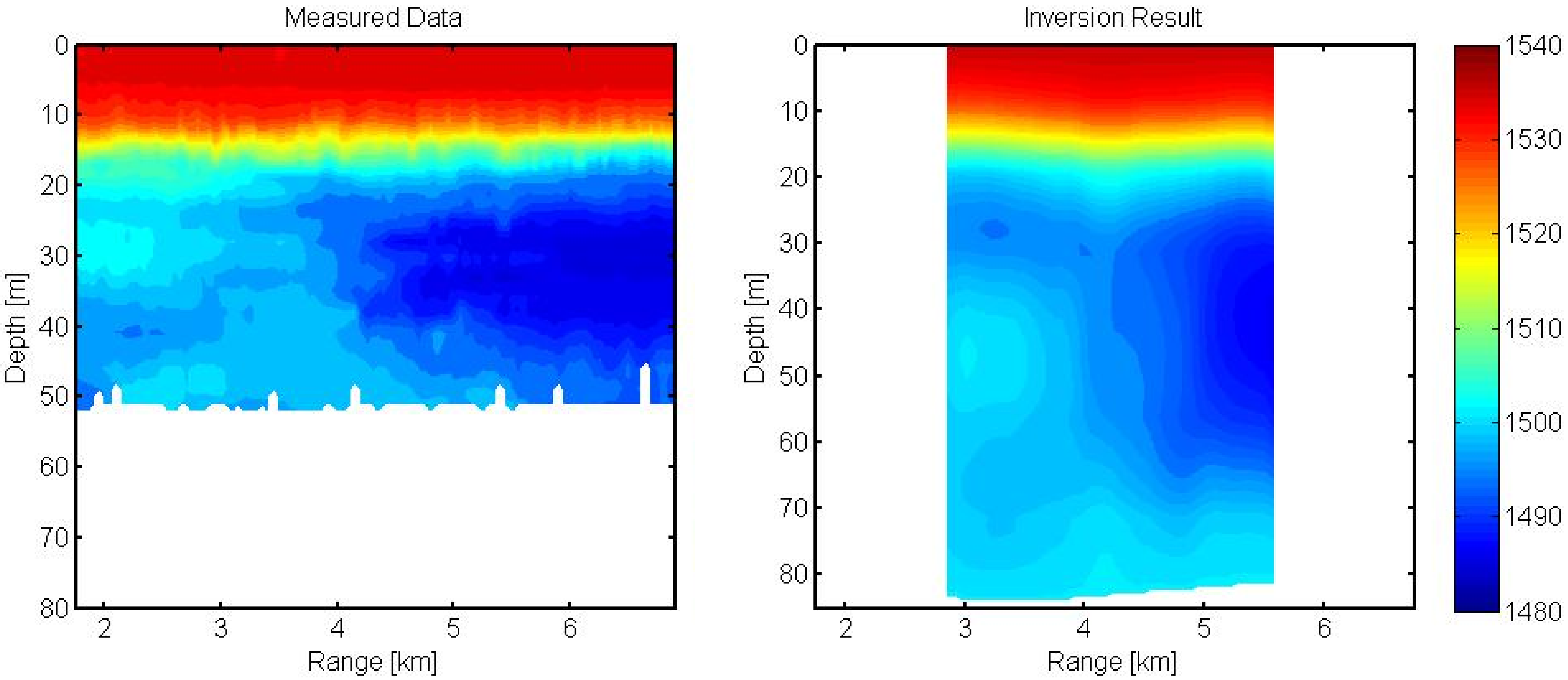
Wave numbers estimated from 125 Hz pressure field
AR estimator using 1600 meter sliding window with 95% overlap



Data from SW06

Results obtained using 125 and 175 Hz wavenumbers

Using low order wavenumbers only.



Conclusions

- Presented an inversion technique for range dependent water column sound speed profiles
  - Robust to inaccurate data and poor knowledge of bottom properties
- Achieved using perturbative inversion with approximate equality constraints
  - The solution is a weighted sum of the data and relative and absolute constraints
- Demonstrated that inversion results could be improved by using low order wave numbers when knowledge of the bottom was poor
  - Effectively separating the water column inversion problem from the sediment inversion problem
- Technique was applied to data from the SW06 experiment
  - Demonstrates the method works on real data