Unperturbed normal mode method for forward surface scattering

Including water-sediment interface

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Unperturbed modes

rather than local modes

- The perturbed ∂_x P only depends on P near the interface.

  This leads to a low-rank coupling matrix.

- Modes only needed to be calculated once.

Variational principle: S is an extremum

$$S = \int L\, dx = \int dx\, dz\, \frac{\left|\partial_x P\right|^2 + \left|\partial_z P\right|^2 - \left|\frac{\omega}{c} P\right|^2}{2\rho}$$

What is the change ΔL from moving an interface up by an amount $h(x)$?

P, $\partial_x P$, and $\partial_z P / \rho$ are continuous across the interface

$$\Delta L = \tfrac{1}{2} h(x) \left(\left|\partial_x P_i\right|^2 \Delta\left(\frac{1}{\rho}\right) + \left|\frac{\partial_z P_i}{\rho}\right|^2 \Delta\rho - \left|\omega\, P_i\right|^2 \Delta\left(\frac{1}{c^2 \rho}\right) \right) + O(h^2)$$

The subscript *i* means that the pressure & its derivatives are evaluated on the interface

Surface of the Ocean

With the approximation that the density of air is zero

$$\Delta\left(\frac{1}{\rho}\right) \text{ and } \Delta\left(\frac{1}{c^2\rho}\right) = \infty$$

Therefore P and ∂_x P vanish, so only the $\Delta\rho$ term survives

$$\Delta L = \tfrac{1}{2}\, h(x)\, \frac{\left|\partial_z P_i\right|^2}{\rho}$$

Mode representation

$$P(x,z) = \sum \phi_j(z)\, a_j(x)$$

$$\int \frac{\phi_j(z)\phi_k(z)}{\rho}\, dz = \delta_{jk}$$

$$L = \tfrac{1}{2} \sum \left(\left|\partial_x a_j\right|^2 - \left|k_j a_j\right|^2 \right) + \Delta L$$

k_j is the horizontal wavenumber of the j'th mode

Reduction to one-way equation

$$\left|\partial_x a_j\right|^2 = \left|(\partial_x - i k_j)^2 a_j\right|^2 - 2\mathrm{Re}\left(i k_j a_j^* (\partial_x - i k_j) a_j\right) + \left|k_j a_j\right|^2$$

(A similar equation holds for cross terms)

First term = $O(h^2)$; dropped $\Rightarrow$ one way equation
Second term = $O(h)$; dropped from ΔL but not L
Third term = $O(1)$; retained in both

Equation for mode amplitudes

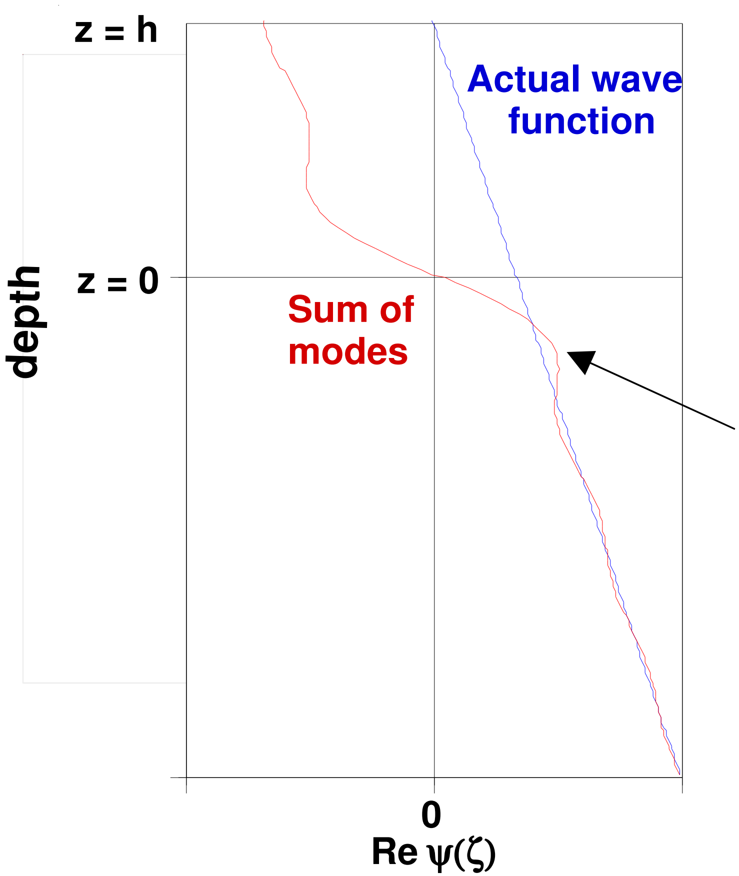
$$\partial_x a_j = i k_j a_j + \frac{i h(x)}{2k_j} \sum_l \gamma_{jl} a_l$$

$$\gamma_{jl} = k_j \varphi_j(z_i)\,\Delta\left(\frac{1}{\rho}\right) k_l \varphi_l(z_i) + \frac{\partial_z \varphi_j(z_i)}{\rho}\,\Delta\rho\,\frac{\partial_z \varphi_l(z_i)}{\rho}$$

$$- \omega \varphi_j(z_i)\,\Delta\left(\frac{1}{c^2 \rho}\right) \omega \varphi_l(z_i)$$

γ Is rank 3 (rank 1 for surface)

Gibbs phenomenon at water surface



Difference = si function

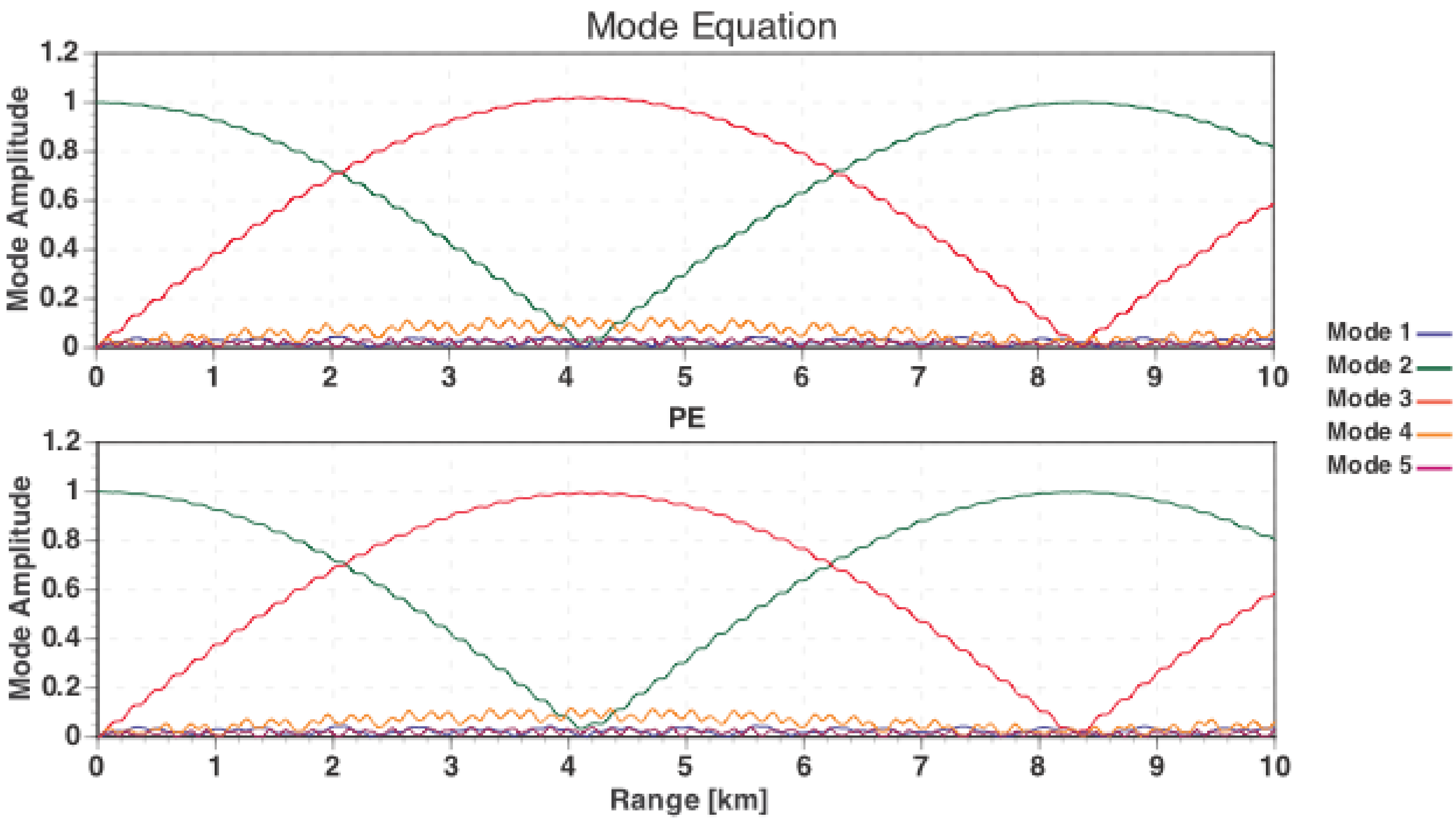
Correct the slope at z = 0 accounting for this si function

Instead of using the slope of the sum of modes at the surface, use it at the depth where the si function has zero slope.

Sine Surface Resonant Case

Wavenumber of the surface = difference of
mode 2 and mode 3 wavenumbers

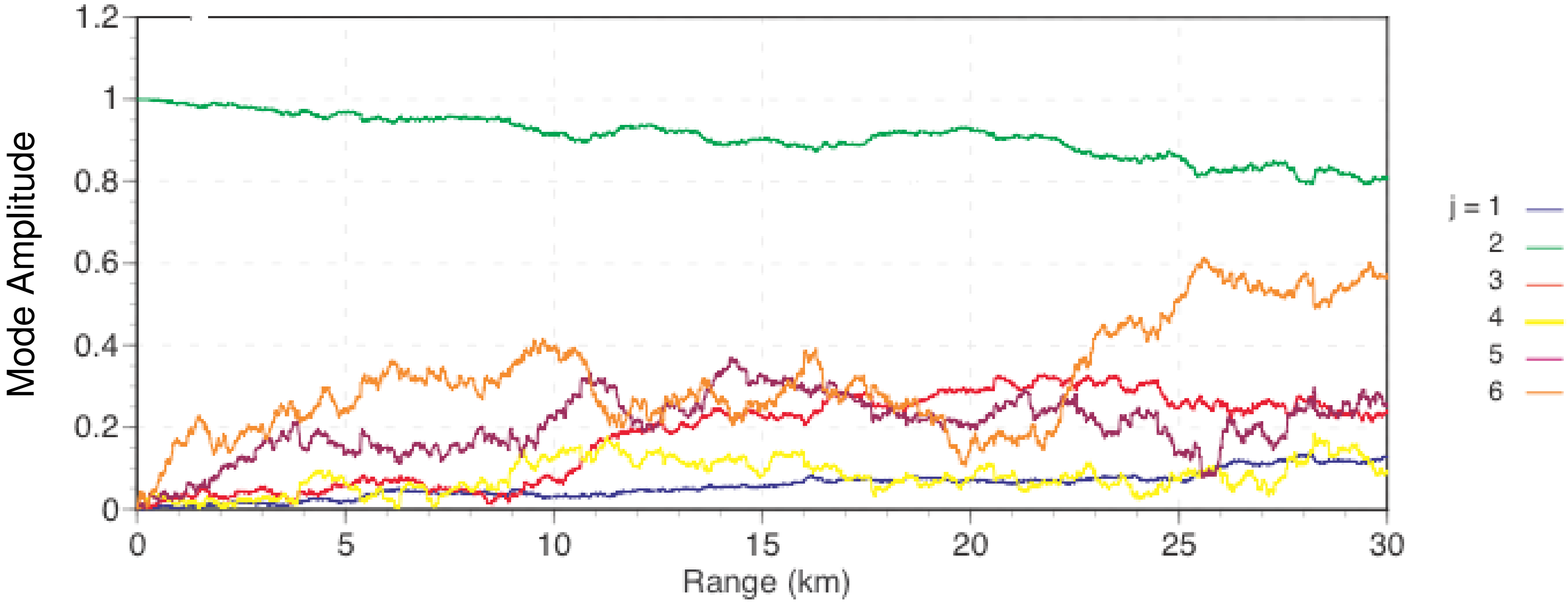
f = 100 Hz, depth = 50 m
Initially, pure mode 2
Isovelocity profile



Realization of a random surface wave field

Pierson-Moskowitz spectrum for 7.7 m/s wind speed

100 Hz, depth = 50 m. Initially mode 2, isovelocity profile



Statistical Moments

Special case of well-known results

Van Kampen, N. *Stochastic Processes in Physics and Chemistry*, 1992, Chapter XVI

Dozier, L. and F. Tappert, "Statistics of normal mode amplitudes in a random ocean. II. Computations," JASA **64**, 533-547 (1978).

First Moment = Coherent Field

Second Moment = Intensity Transport

Matrix Formalism

$$\partial_x \mathbf{A} = i\mathbf{K}\mathbf{A} + ih(x)\mathbf{M}\mathbf{A}$$

A is a vector of mode amplitudes

K is a diagonal matrix of horizontal wavenumbers

$h(x)$ is a stationary random process with correlation function

$$C(x) = \langle h(x)\, h(x+x) \rangle$$

M is the coupling matrix

Define a new matrix of correlated shorter-range couplings

$$\mathbf{N} = \int_0^x C(x) \exp(i\mathbf{K}x)\mathbf{M}\exp(-i\mathbf{K}x)\,dx$$

The integrand is low rank, but the integral is not

Moment equations

First moment equation (lowest order in h)

$$\partial_x \langle \mathbf{A} \rangle = i\mathbf{K}\langle \mathbf{A} \rangle - \mathbf{MN}\langle \mathbf{A} \rangle$$

Second moment (Transport) equation

Use outer product $\langle \mathbf{A}\,\mathbf{A}^{+} \rangle$ Mode intensities are its diagonal elements.

$$\partial_x \langle \mathbf{A}\,\mathbf{A}^{+} \rangle = i\mathbf{K}\langle \mathbf{A}\,\mathbf{A}^{+} \rangle - \mathbf{MN}\langle \mathbf{A}\,\mathbf{A}^{+} \rangle - i\langle \mathbf{A}\,\mathbf{A}^{+} \rangle\mathbf{K} - \langle \mathbf{A}\,\mathbf{A}^{+} \rangle\mathbf{N}^{+}\mathbf{M}^{+}$$
$$+\mathbf{N}\langle \mathbf{A}\,\mathbf{A}^{+} \rangle\mathbf{M}^{+} + \mathbf{M}\langle \mathbf{A}\,\mathbf{A}^{+} \rangle\mathbf{N}^{+}$$

This form only works because there is a single random function h.

[Morozov and Colosi, eq. 16]

Simplifications

Dozier & Tappert conjecture:

Off-diagonal components of $\langle AA^{+} \rangle$ are small, and can be neglected.

This conjecture is found to not be true.

Replacement conjecture:

$\langle AA^{+} \rangle \approx \langle A \rangle \langle A^{+} \rangle$ + diagonal

By assuming $AA^{+} - \langle A \rangle \langle A^{+} \rangle$ is a Gaussian process, we can calculate all moments, such as the mode scintillation index

Results (Water Surface)

For all results that follow:

compared to Collins PE with surface effects added by Rosenberg

flat sea floor, absorbing bottom (0.5 dB/λ)

isovelocity waveguide

the frequency is 3 kHz

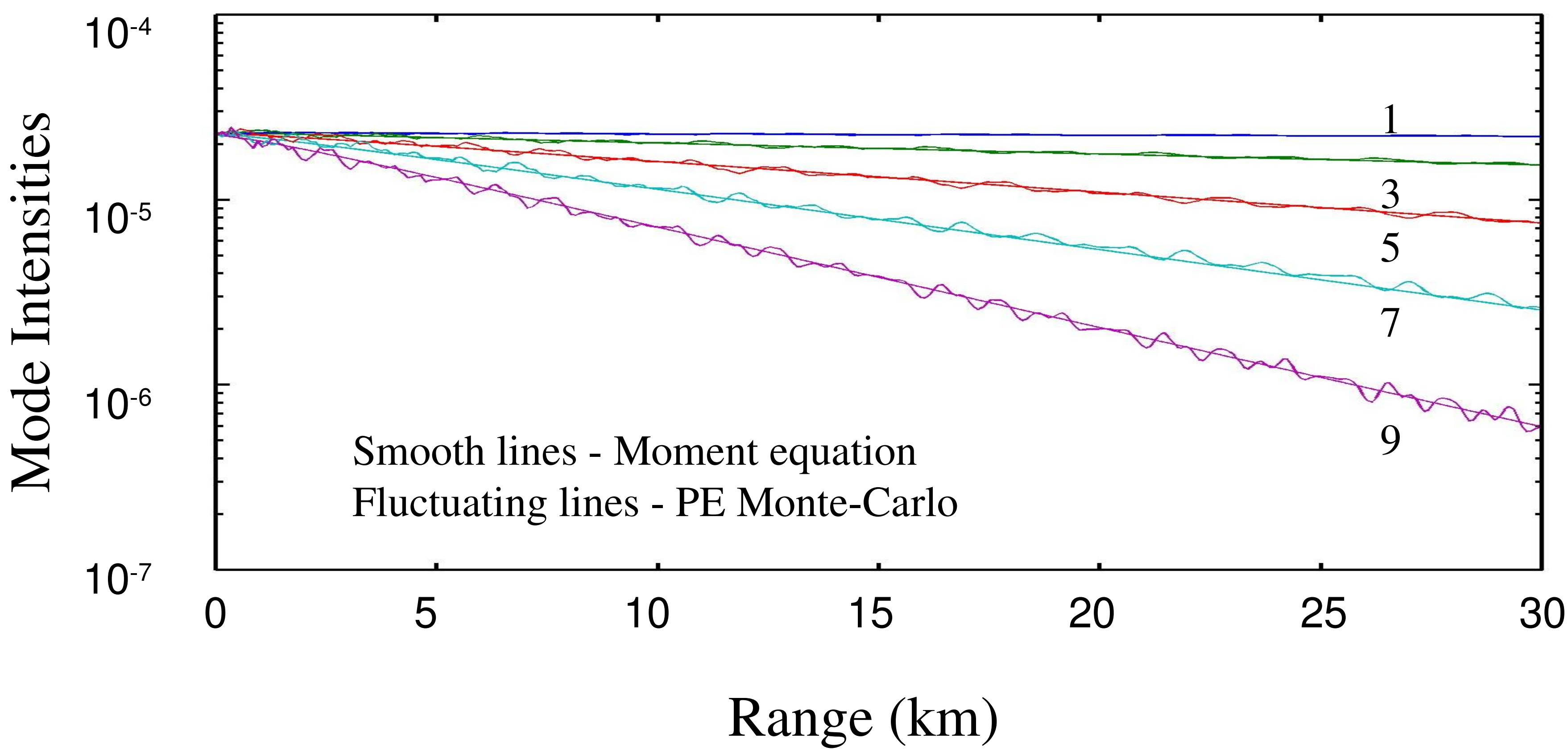
Pierson-Moskowitz wind wave spectrum (15 knots)

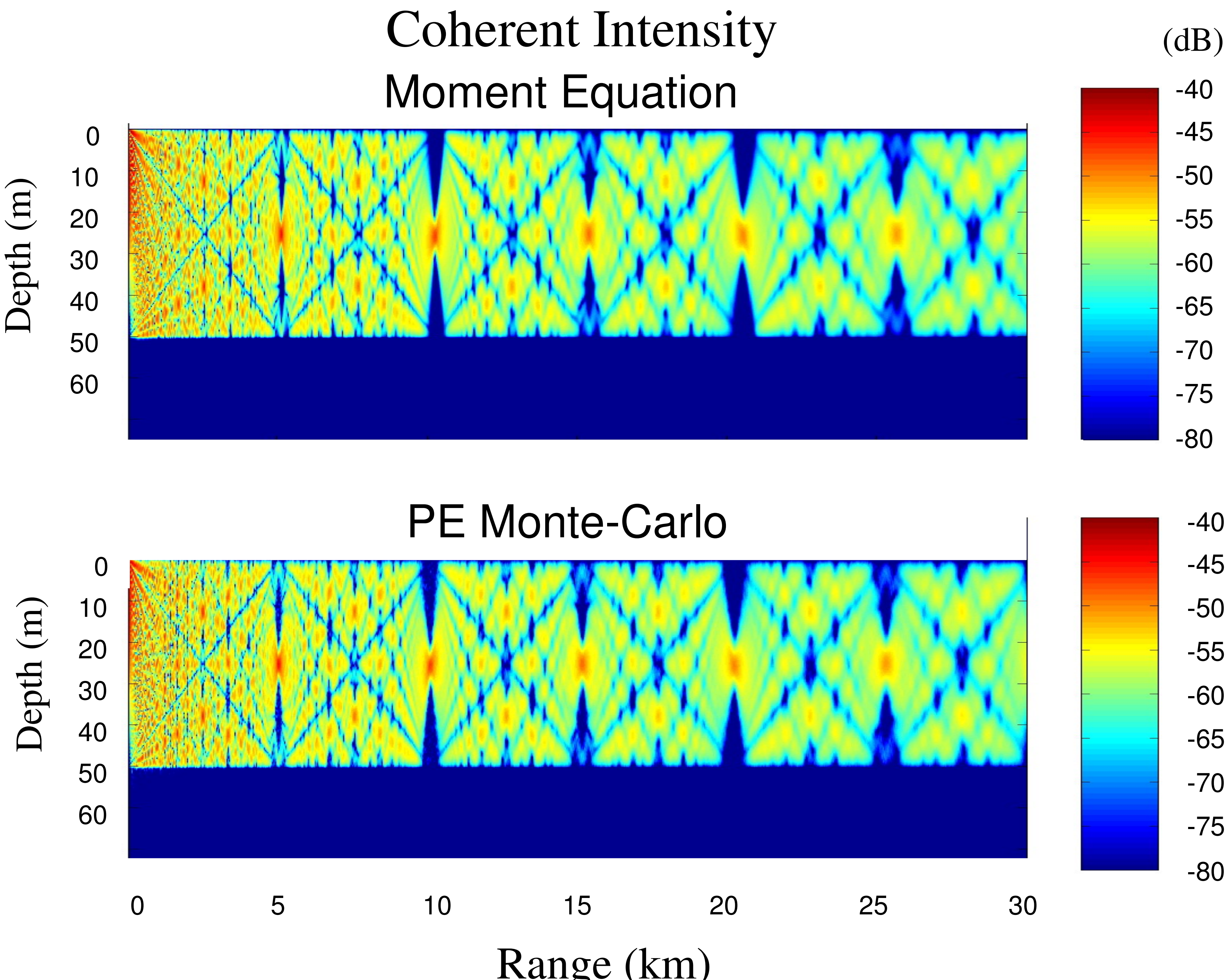
the water depth is 50 m

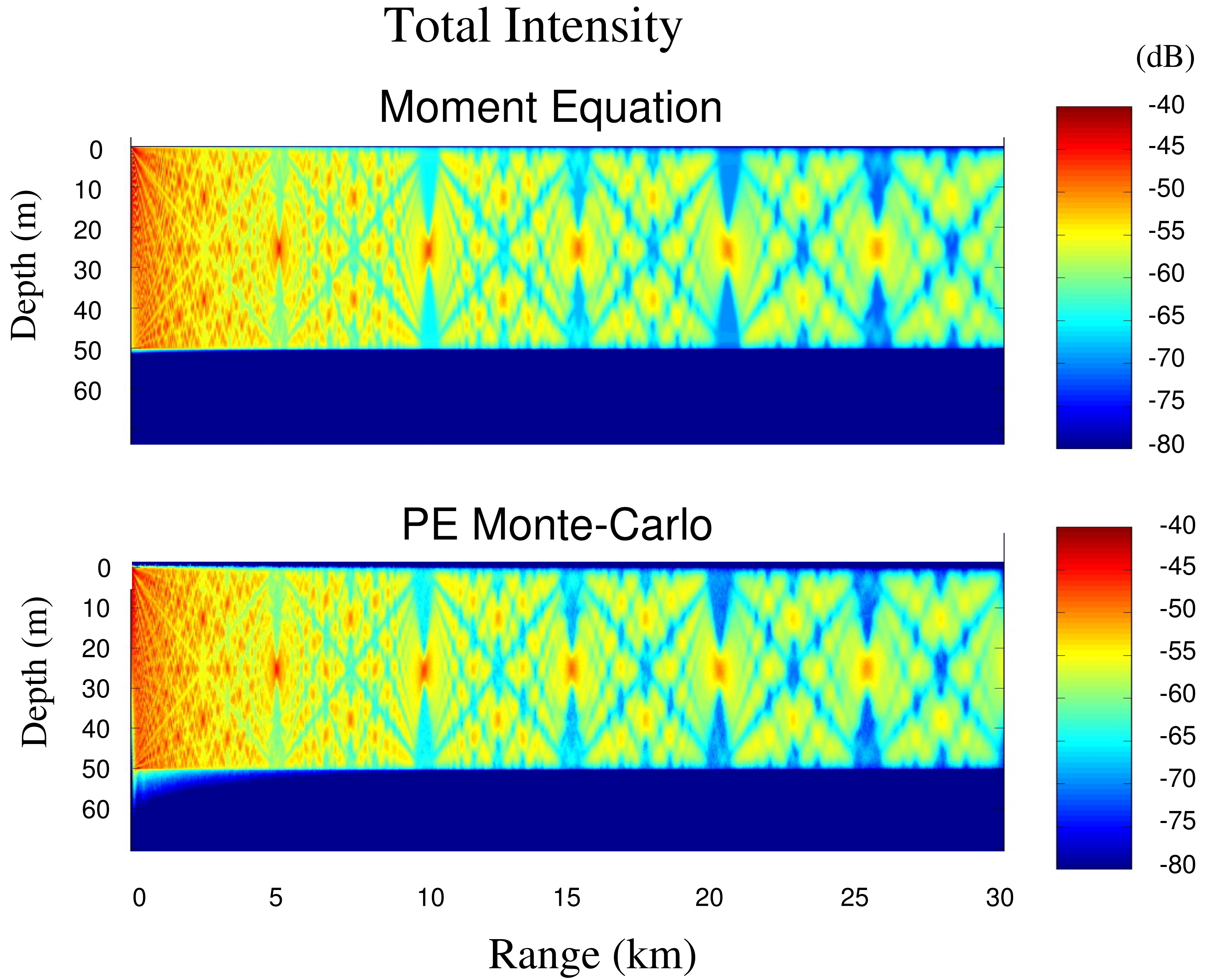
source at 25 m

moment equation results use “replacement conjecture”

Mode Intensities (odd mode numbers)




Coherent Intensity
Moment Equation
PE Monte-Carlo
Depth (m)
0
10
20
30
40
50
60
Range (km)
0
5
10
15
20
25
30
(dB)
-40
-45
-50
-55
-60
-65
-70
-75
-80


Total Intensity
(dB)
Moment Equation
PE Monte-Carlo
Depth (m)
0
10
20
30
40
50
60
-40
-45
-50
-55
-60
-65
-70
-75
-80
0
5
10
15
20
25
30
Range (km)

Conclusions

Unperturbed modes work well for surface and (we expect) bottom roughness, both deterministically and stochastically, when the mean waveguide is range independent.

Mismatch of boundary conditions between unperturbed and actual surfaces can be corrected for.

The intensity transport equation can be written with only $N \times N$ matrices, avoiding further approximation.

The Dozier-Tappert conjecture doesn't work for boundary scattering, but assuming it for only the incoherent field works in the examples we tried.

Even the scintillation index can be calculated using this conjecture, assuming Gaussian statistics for the incoherent field.